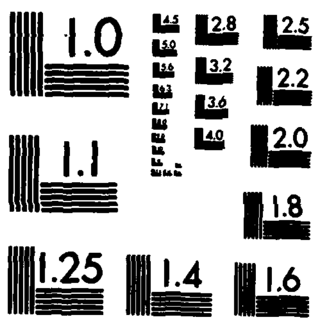
AD-A124 459 NONLINEAR FILTERING USING LINEAR COMBINATIONS OF ORDER STATISTICS(U) ILLINOIS UNIV AT URBANA COORDINATED SCIENCE LAB A C BOVIK JAN 82 R-935 N00014-79-C-0424 1/2

UNCLASSIFIED F/G 12/1 NL



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

AD A124459

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER	2. GOVT ACCESSION NO. AD-A124 459	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) NONLINEAR FILTERING USING LINEAR COMBINATIONS OF ORDER STATISTICS		5. TYPE OF REPORT & PERIOD COVERED Technical Report
		6. PERFORMING ORG. REPORT NUMBER R-935; UILU-ENG 82-2201
7. AUTHOR(s) ALAN CONRAD BOVIK		8. CONTRACT OR GRANT NUMBER(s) N00014-79-C-0424(25)
9. PERFORMING ORGANIZATION NAME AND ADDRESS		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS Coordinated Science Laboratory University of Illinois at Urbana-Champaign Urbana, Illinois 61801		12. REPORT DATE January 1982
		13. NUMBER OF PAGES 120
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office) Joint Services Electronics Program		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

Approved for public release; distribution unlimited.

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

A nonlinear filter is considered whose output is given by a linear combination of the order statistics of the input samples. Assuming a constant signal in white noise, the coefficients are chosen to minimize the output MSE for several noise distributions. This new general filter is superior to both the median filter and the averaging filter, since they are just special cases.

A class of heavy-tailed noise distributions is then considered, and optimal coefficients are found for various parameter values to determine the dependence of the filter coefficients on the heavy-tailedness (impulsivity) of the additive

DD FORM 1 JAN 73 1473

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

Cont

noise.

Finally, some examples of designed filters are given operating on some simple inputs, and an examination of the root sets of the general filter is given.

DTIC COPY INSPECTED 2

Accession For
NTIS GRA&I
DTIC TAB
Unannounced
Justification

By
Distribution/
Availability Codes
Avail and/or
Dist Special
A

NONLINEAR FILTERING USING LINEAR
COMBINATIONS OF ORDER STATISTICS

BY

ALAN CONRAD BOVIK

B.S., University of Illinois, 1980

THESIS

Submitted in partial fulfillment of the requirements
for the degree of Master of Science in Electrical
Engineering in the Graduate College of the
University of Illinois at Urbana-Champaign, 1982

Urbana, Illinois

ACKNOWLEDGEMENTS

I would like to extend my deep appreciation to Dr. David C. Munson and Dr. Thomas S. Huang for their guidance and support. Without their helpful suggestions and contributions, this work could not have been done.

TABLE OF CONTENTS

I. INTRODUCTION

Median filtering has recently been recognized as an effective alternative to the linear smoother for noise suppression. In particular, the moving median of a time or spatial series has been shown to preserve edges or monotonic changes in trend, while eliminating impulses of short duration. In these respects, the median smoother is superior to the linear filter which tends to blur edges and smear impulses, which may not be desireable in certain applications.

The median is a particular case of the ith order statistic (or rank statistic) of a finite set of real numbers. The ith order statistic of N real numbers $x_1,\ldots,x_N$ where N is usually odd for digital filtering applications, is defined as the ith largest number in algebraic value. Here we shall denote the ith order statistic by $x_{(i)}$ in keeping with the mathematical literature. The minimum is then $x_{(1)}$, the maximum $x_{(N)}$, and the median $x_{((N+1)/2)}$.

Little work has been done in digital filtering applications using order statistics other than the median. It is the object of this thesis to present a general order statistic filter design scheme, where the output of the

filter is a linear combination of the order statistics of several input samples considered simultaneously. The order statistic filter is nonlinear due to the ordering process, which considerably complicates the analysis. Thus, several assumptions are made to simplify the mathematics. First, the original (uncorrupted) input sequence is assumed to be constant throughout; further, the additive noise is assumed to be zero-mean and white.

In section II, the optimal coefficients of the order statistics under minimization of the mean-squared error (MSE) are found for several symmetric noise distributions. This is accomplished by first calculating the correlation matrices of the order statistics for each noise distribution. The minimization problem is then treated as a classical quadratic problem, and a solution is found using Lagrange multipliers. This procedure is done for all odd filter lengths ranging from three to twenty-five.

As mentioned above, the median filter has been noted to be effective in suppressing impulsive or heavy-tailed noise. In section III a class of heavy-tailed noise distributions is considered, and optimal coefficients are again found, for a filter length of three. The resulting coefficients give considerable support to the notion of the median filter as a useful tool in suppressing heavy-tailed noise.

Finally, in section IV some deterministic (non-statistical) properties of the class of filters under consideration are examined. A computer simulation of some of the filters designed in section II is done for some simple input structures, particularly those involving edge- and impulsive-type shapes. We conclude the section with a discussion regarding the description of the set of input signals to which a given filter is invariant.

In order to provide a suitable impetus for the work that will follow, a short review of the previous work done in median filtering follows.

A. Early Developments in Median Filtering

The median filter was first proposed in 1971 by Tukey [1], who used the moving median as a smoothing technique in economic time series filtering problems. Rabiner et al. [2] used the median filter and series combinations of linear filters and median filters to smooth speech waveforms in a qualitative study, and reported favorable results. They found that the median filter was generally superior to a Hanning window of similiar length for smoothing several waveforms such as log input energy of a speech signal, zero-crossing rate, and pitch period. The median smoother was noted to preserve discontinuities of sufficient duration while eliminating local roughness in the signal, whereas the

linear smoother was seen to be inadaquate in that much information was lost due to smearing. They deemed the median-linear series combination to be yet more effective for their application, but there is no conclusive evidence to support this for the general application.

Jayant [3] performed a similiar study involving the suppression of impulsive noise due to bit errors in the transmission of digital speech signals. He compared moving average- and moving median-based filters in computer simulations and informal listening tests for various filter lengths. He concluded that for independently occuring errors the two techniques performed similiarly. However, he noted that for dependent error occurences, which were interpreted as clusterings of the errors, the averager was generally superior, and longer filter lengths for both filters were more effective.

These are intuitive conclusions since a cluster of similar impulses may be interpreted as an edge or change of trend by a median filter, which may preserve it if the cluster is sufficiently long.

Median filtering has also come into recent use for enhancing images. Pratt [4] made a rather qualitative study of two dimensional median filters of various sizes and shapes, and examined application of one dimensional medians to a picture corrupted by impulsive noise. He concluded that although the median filter is extremely useful for

suppressing impulsive and "salt-and-pepper" noise, it should be considered an ad hoc method dependent upon the particular application.

There has been some more recent work in actual implementation of median filtering. Huang et al. [5] have devised a fast algorithm to implement two-dimensional median filters. It is based upon the fact that as the filter window is moved from column to column across the image, most of the pixels are retained within the filter window. The algorithm updates the histogram corresponding to the new pixel values entering the window. A considerable reduction in computer time is attainable when compared with conventional sorting algorithms. In particular, for an mxn filter window the number of comparisons required for computation of each pixel is about (2n+10); using ordinary sorting methods the number of comparisons is approximately mn.

An algorithm for real-time median filtering has been developed by Ataman, Aatre, and Wong [6]. This algorithm allows on-line computation of the running median.

It is possible that some of these methods or variations could be used for general order statistical filtering; however that is beyond the scope of this thesis, and will not be discussed here.

B. Recent Developments and Preliminary Definitions

More recently, several authors have begun a more rigorous study of median-based and other order statistic-related filtering methods. Many of the results they have attained are of relevance to the work here, and we will develop some of our definitions and objectives as we examine what has already been done. We will begin with some definitions.

An order statistical filter of length N operating on a sequence $\{x_j;\ j \text{ an integer}\}$ for N assumed to be odd will be defined as

$$y_j = OSF(x_j) = \sum_{i=1}^{N} \alpha_i x_{(i)} \qquad (1)$$

where the $x_{(i)}$ are the order statistics of $x_{j-M},\ldots,x_j,\ldots,x_{j+M}$ with $M = (N-1)/2$. The α_i are real constants upon which we may impose certain restrictions, as we will do later. We can generalize this to two dimensions by noting that we merely consider the points within the window as the values to be ranked and linearly combined, regardless of the shape and size of the window. Generally the window is symmetric about its center and we replace the center pixel value with the output value.

The median filter is a particular case of (1), where we define the α_i as

$$\alpha_i = 0 \ ; \ i = 1,\ldots,N; \ i \neq (N+1)/2$$

$$\alpha_i = 1 \ ; \ i = (N+1)/2 \tag{2}$$

or

$$y_j = MF(x_j) = x_{((N+1)/2)}. \tag{3}$$

We can also define a maximum filter, for example, by defining

$$\alpha_i = 0 \ ; \ i = 1,\ldots,N-1$$

$$\alpha_i = 1 \ ; \ i = N \tag{4}$$

or

$$y_j = MAXF(x_j) = x_{(N)}. \tag{5}$$

In general, we may constrain all of the coefficients of the order statistics to be zero except for one, which we set to unity. We say that if we defined α_i to be unity, we have an ith ranked-order operation. As an example of an application, Nodes and Gallagher [7] have found that an (N-1)th ranked-order operation is effective for digital AM detection with and without corruption by impulsive noise.

Similiarly, an Nth ranked-order operation can be used for peak detection.

For clarity, let us consider a simple input signal operated on by some simple filters of length three and five. It should be noted first that in considering input signals that are either of finite length or are causal we have a difficulty in defining the filtering operation for the $M = (N-1)/2$ points closest to the endpoint(s) of the input signal. In keeping with the conventions used in previous literature, we will remedy this situation by appending M points to the left or right of each endpoint, each of the same value as the input sample value at the endpoint.

Consider the simple input signal x_n shown in Figure 1. In Figure 2 are shown the outputs y_n after one pass of three filters, each of length three; a median filter, an averaging filter, and a midpoint filter which are described by $(\alpha_1,\alpha_2,\alpha_3) = (0,1,0)$, $(1/3,1/3,1/3)$, and $(1/2,0,1/2)$, respectively. Figure 3 shows the outputs y_n after a single pass of the same filters but for a filter length of five; hence $(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5) = (0,0,1,0,0)$, $(1/5,1/5,1/5,1/5,1/5)$, and $(1/2,0,0,0,1/2)$, respectively.

It is apparent that different filter coefficients result in varied filter outputs, and that filters of a definite class but different lengths will give different outputs for identical input signals. This will be examined at greater length in section IV; however it is worth noting

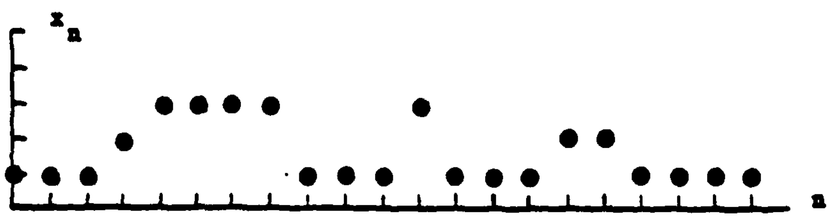


Figure 1. Original simple input signal

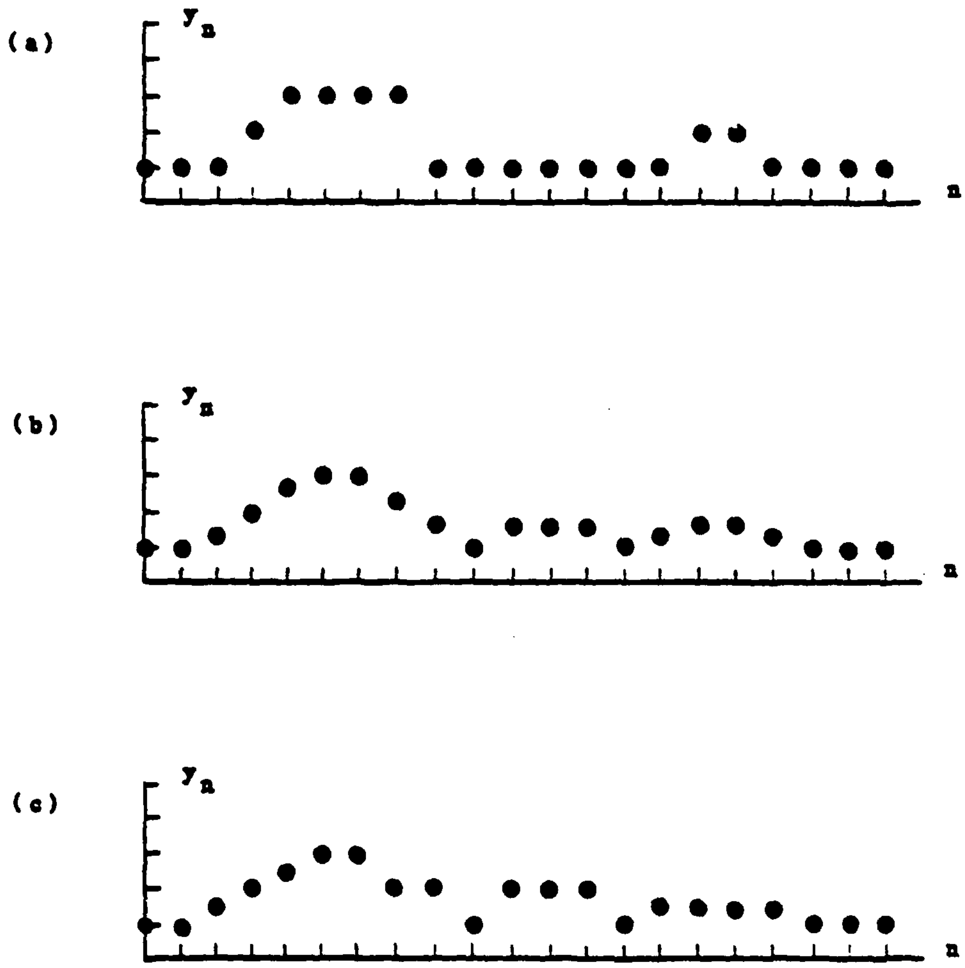


Figure 2. Signal filtered by three different filters, N=3
(a) Median filter
(b) Averaging filter
(c) Midpoint filter

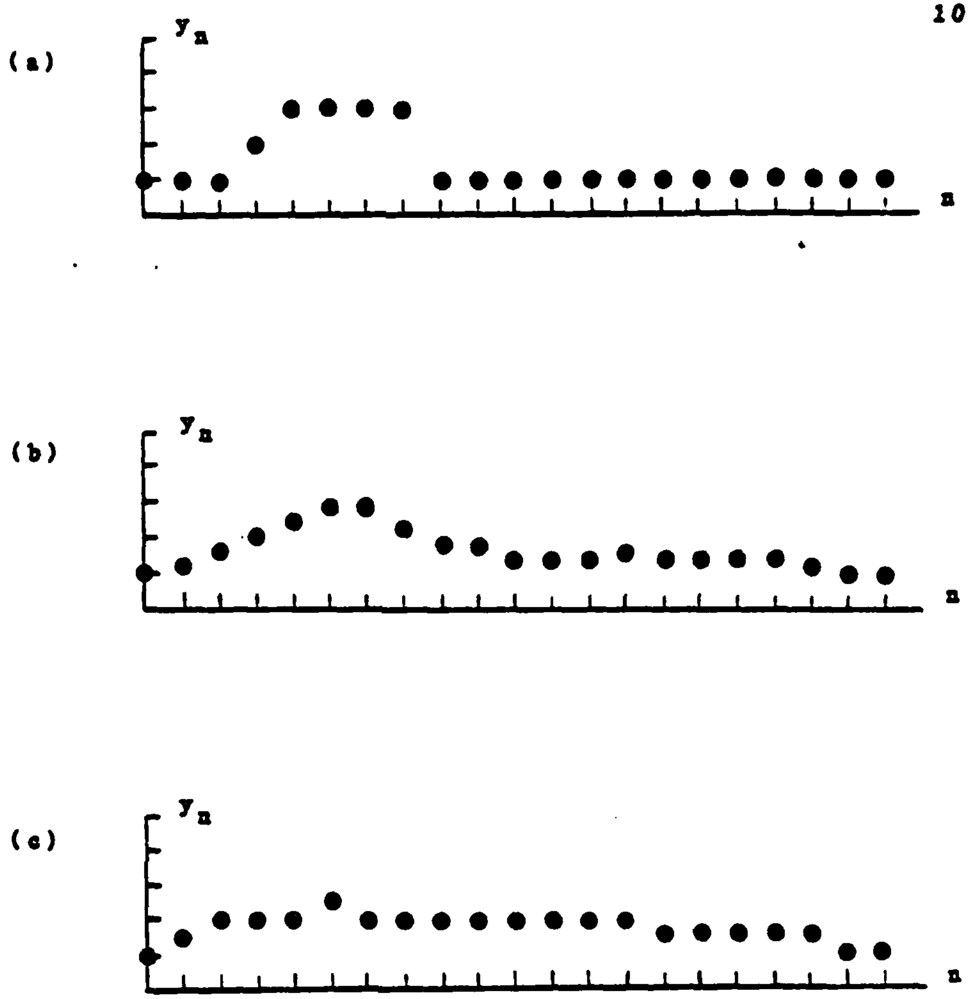


Figure 3. Signal filtered by three different filters, N=5
(a) Median filter
(b) Averaging filter
(c) Midpoint filter

at this point that the median filters preserve edges of sufficiently long duration and eliminate impulses of sufficiently short duration, but that no such rule applies to the other filters.

II. OPTIMAL FILTER COEFFICIENTS FOR SEVERAL SYMMETRIC NOISE DISTRIBUTIONS

Our objective is to find values for the coefficients α_i subject to certain optimality criteria and other specific constraints. For simplicity, we consider a constant signal level of value s corrupted by zero-mean white noise; thus the input sample values are of the form

$$x_j = s + n_j \tag{6}$$

where the n_j are independent identically distributed (IID) random variables satisfying

$$E[n_j] = 0 \text{ ; } j \text{ an integer} \tag{7}$$

and hence

$$E[x_j] = s \text{ ; } j \text{ an integer.} \tag{8}$$

As an optimality condition we shall consider minimization of the MSE, but we would like to insure that the order statistical estimator that we design is unbiased, or independent of the value of the constant background value, s.

The MSE is given by

$$MSE = E[(y_j - s)^2] = E\left[\left(\sum_{i=1}^{N} \alpha_i x_{(i)} - s\right)^2\right]. \qquad (9)$$

Using (6), we see that

$$x_{(j)} = s + n_{(j)} \qquad (10)$$

where the $n_{(j)}$ are the order statistics of the additive noise corrupting the original constant signal. We then have

$$MSE = E\left[\left(\sum_{i=1}^{N} \alpha_i n_{(i)} + s\sum_{i=1}^{N} \alpha_i - s\right)^2\right] \qquad (11)$$

It is apparent that if we are to have an unbiased estimator we must require that

$$\sum_{i=1}^{N} \alpha_i = 1. \qquad (12)$$

In this case our problem reduces to minimization of

$$MSE = E\left[\left(\sum_{i=1}^{N} \alpha_i n_{(i)}\right)^2\right] \qquad (13)$$

In a related study, Justusson [8] used asymptotic formulae to find approximate expressions for the first and second moments of the $x_{(j)}$ for several parent noise distributions, in terms of N. He found that for heavy-tailed distributions such as the double exponential (Laplacian), the moving median was significantly superior to the moving average filter in estimating the constant signal, s. In fact, the median is the maximum-likelihood-estimator

(MLE) of s for the Laplacian distribution in an asymptotic sense (as $N\uparrow\infty$). It is also worth noting that the Laplacian distribution is often used to model impulsive-type noise, which the median is well known to suppress. However, the averaging filter was superior to the median filter for lighter-tailed parent distributions such as the uniform and normal distributions. Again, this is an expected result since the average is the MLE of the mean for the normal distribution [9].

However, Justusson then applied moving medians and averages to an image corrupted by normal noise $\eta(0,\sigma)$ for various values of the ratio $R = \sigma/h$, where h is the largest edge differential in the picture. He found that for reasonably small values of R, the median performed at least as well as the average in reconstructing the image, with sharp edges. Following this observation, he did an analytic comparison of the behavior of averages and medians across a noise-corrupted edge, in one dimension. To compare the relative efficiencies of the filters on an edge-plus-noise sequence he used the average of the MSEs at K points close to the edge:

$$\sum_j E[(y_j - s_j)^2]/K \qquad (14)$$

where the s_j are equal to a constant value s on one side of the edge, and are equal to a constant value s+h on the other, where h is any real number representing the height of

the edge. Using the values N=3, K=2 he found that for R > 1/2 the averager gave very slightly better results than the median, but for R < 1/3 the median was far better.

The criteria used in (14) is a very interesting one, since we could insure sharpness about an edge using it in a filter design procedure. The problem in such a procedure is that we could not have an unbiased estimator; we would have to design using some a priori knowledge about R. Further, we would be dealing with two separate parent distributions corresponding to each side of the edge; this immensely complicates the analysis, although it does not render it untenable. However, we will restrict ourselves to the constant background assumption in this *thesis*. We will now attack the problem of computing the correlation matrices in preparation for the minimization procedure.

A. Computation of the Correlation Matrices of the Order Statistics of the Additive Noise

We begin with

$$MSE = \sum_{i=1}^{N} a_i^2 E[n_{(i)}^2] + \sum_{i=1}^{N} \sum_{j=1,\, j\neq i}^{N} a_i a_j E[n_{(i)} n_{(j)}] \qquad (15)$$

where we have split the MSE into two terms so that we need only calculate the second moments and the correlations of

the $n_{(j)}$, $j = 1,\ldots,N$. To compute these, we first need expressions for the densities and joint densities of the $n_{(j)}$. For this, we can refer to the excellent monograph on order statistics by David [9]. If we denote the parent distribution and density of the noise as $F_n(\cdot)$ and $f_n(\cdot)$ respectively, then the density of $n_{(j)}$ for $j = 1,\ldots,N$ is given by

$$g_{n_{(j)}}(x) = K_j F_n^{j-1}(x)[1-F_n(x)]^{N-j} f_n(x)$$

$$\text{where } K_j = \left[N!/[(j-1)!(N-j)!]\right] \qquad (16)$$

and for the joint density of $n_{(i)}$ and $n_{(j)}$ where $i,j = 1,\ldots,N$ $(i<j)$ we have

$$g_{n_{(i)}n_{(j)}}(x,y) = K_{i,j} F_n^{i-1}(x)[F_n(y)-F_n(x)]^{j-i-1} \cdot [1-F_n(y)]^{N-j} f_n(x) f_n(y)$$

$$\text{where } K_{i,j} = \left[N!/[(i-1)!(j-i-1)!(N-j)!\right]. \qquad (17)$$

A derivation of these densities is given in David, Chapter 2. We now have

$$\begin{aligned} MSE &= \sum_{i=1}^{N} \alpha_i^2 \int_{-\infty}^{\infty} x^2 g_{n_{(i)}}(x)\, dx \\ &+ \sum_{i=1}^{N-1}\sum_{j=i+1}^{N} \alpha_i \alpha_j \int_{-\infty}^{\infty}\int_{-\infty}^{y} xy\, g_{n_{(i)}n_{(j)}}(x,y)\, dxdy \\ &+ \sum_{i=2}^{N}\sum_{j=1}^{i-1} \alpha_i \alpha_j \int_{-\infty}^{\infty}\int_{-\infty}^{y} xy\, g_{n_{(j)}n_{(i)}}(x,y)\, dxdy \qquad (18) \end{aligned}$$

Note that the positions of i and j are opposite in the second and third terms of (18), since the second term gives the correlations for $i < j$ whereas the third term gives the correlations for $j < i$.

Numerical integration is necessary in general for the computation of the integrals in (18). However, the number of computations required can be reduced by making use of the following three relations, which hold when the parent distribution is symmetric about the origin (of course, the first equality in (19) is always true):

$$
\begin{aligned}
E[n_{(i)}n_{(j)}] &= E[n_{(j)}n_{(i)}] \\
E[n_{(j)}] &= -E[n_{(N-j+1)}] \qquad (19) \\
E[n_{(i)}n_{(j)}] &= E[n_{(N-j+1)}n_{(N-i+1)}]
\end{aligned}
$$

Evaluation of (18) gives an expression for the MSE of the form

$$MSE = \sum_{i=1}^{N}\sum_{j=1}^{N} \alpha_i \alpha_j H_{ij} \qquad (20)$$

where the evaluated integrals in (18) are computed to give the correlations $H_{ij} = E[n_i n_j]$. We can express this as in quadratic form as

$$MSE = \underline{\alpha}^T H \underline{\alpha} \qquad (21)$$

where H is the NxN correlation matrix of the random vector

$(x_{(1)},\ldots,x_{(N)})^T$ and $\underline{\alpha}$ is the constant vector $(\alpha_1,\ldots,\alpha_N)^T$. Using the symmetry relations in (19), we find that instead of computing N single integrations and N(N-1) double integrations (N^2 total) we need only compute (N+1)/2 single integrations and $(N^2-1)/4$ double integrations ((N+1)/4 total). We see that for large N we nearly quarter the number of integrations required per matrix.

The correlation matrices for six different parent distributions were calculated for all odd values of the filter length N ranging from three to twenty-five. The parent distributions chosen were the uniform, parabolic, triangular, U-shaped, and normal distributions, as Sarhan [10], [11], [12] did for some small values of N (both odd and even) in estimating population means and variances in the mid-1950's. Sarhan also showed that the problem was independent of σ, so that we can normalize our distributions. The densities we are using are given by (zero-mean):

1. Uniform Density:

$$f(x) = 1/(\sqrt{3}\sigma) \quad ; |x| < \sqrt{3}\sigma$$

$$= 0 \quad ; \text{else} \qquad (22)$$

2. Parabolic Density:

$$f(x) = 3(\sqrt{5}\sigma + x)(\sqrt{5}\sigma - x)/(20\sqrt{5}\sigma^3) \quad ; |x| < \sqrt{5}\sigma$$

$$= 0 \qquad ; \text{else} \qquad (23)$$

3. Triangular Density:

$$f(x) = 2(\sqrt{6}\sigma/2 - |x|)/(3\sigma) \quad ; |x| < \sqrt{6}\sigma/2$$

$$= 0 \qquad ; \text{else} \qquad (24)$$

4. U-Shaped Density:

$$f(x) = \sqrt{27/125}(3x^2/2\sigma^2) \quad ; |x| < \sqrt{5/3}\sigma$$

$$= 0 \qquad ; \text{else} \qquad (25)$$

5. Laplacian Density:

$$f(x) = e^{(-|x|/\sigma)/2\sigma} \quad ; |x| < \infty \qquad (26)$$

6. Normal Density:

$$f(x) = e^{(-x^2/2\sigma^2)}/\sqrt{2\pi}\sigma \quad ; |x| < \infty \qquad (27)$$

These six distributions give a wide range of behavior ranging from the fairly heavy-tailed Laplacian to the very shallow-tailed uniform densities, which should give some indication of the kind of filter we can expect for a given distribution. Note that all six distributions are symmetric

about the origin; other interesting distributions we could have used are the Rayleigh, exponential, and Chi-squared. However, since these are non-symmetric, the second and third relations in (19) do not hold, resulting in much more computation time.

The correlation matrix H for each of the six parent distributions for all odd values of N ranging from three to twenty-five are shown in Tables XIV-XXV respectively, in the Appendix. The elements of each matrix are correct to at least five decimal digits as compared with tables of the lower moments for a normal parent distribution given by Teichrow [13]. The integration routines used in our study were based upon Gaussian quadrature algorithms, which give very precise results for polynomial integrands. With the exception of the Laplacian and normal parents, all of the integrals evaluated had polynomial integrands. The accuracy in the normal case therefore indicates good accuracy overall.

B. Computation of the Optimal Filter Coefficients

From (21), we have again

$$MSE = E[(y-s)^2] = \underline{a}^T H \underline{a} \qquad (28)$$

where s is our original constant background signal value and y is the output of the order statistic filter (OSF) of

length N. Minimization of this quantity is now a straightforward quadratic problem since H is a positive semi-definite symmetric matrix as a consequence of the Schwartz inequality and (19). Therefore (28) represents a convex function, and with the constraint in (12) we can minimize using Lagrange multipliers. The Lagrangian function is then given by (where $\underline{e}$ denotes a column of ones)

$$F(\underline{\alpha},\lambda) = \underline{\alpha}^T H \underline{\alpha} + \lambda(1 - \underline{e}^T\underline{\alpha}). \tag{29}$$

Differentiation then yields

$$2H\underline{\alpha} - \underline{e}\lambda = 0 \tag{30}$$

which combined with (12), which may be written as

$$\underline{e}^T\underline{\alpha} = 1 \tag{31}$$

finally gives

$$\underline{\alpha} = H^{-1}\underline{e}/[\underline{e}^T H^{-1}\underline{e}]. \tag{32}$$

A complete discription of the quadratic problem is given in Bazaraa and Shetty [14], Chapter 11.

The resulting optimal values of the coefficients $\{\alpha_i\ ;\ i = 1,\ldots,N\}$ for each of the six parent distributions and for all odd values of N ranging from three to twenty-five are given in Tables I-XII respectively.

Comparison with Sarhan and Greenberg's results in [15] for N = 3,5 indicate that the values of α_i obtained are accurate to four decimal places.

The results obtained for the uniform and normal parent distributions are expected, since the midpoint ($\alpha_1 = \alpha_N = 1/2$) and the average ($\alpha_i = 1/N$; $i = 1,\ldots,N$) are the MLEs for these respectively. [9] The resulting α_i for the Laplacian parent are not suprising either as we see that most of the weight is located in the center α_i, becoming more pronounced, although more spread among a larger number of center positions, as N increases. The parabolic and triangular results are seen to be similiar to the uniform for small N, but the α_i tend to spread in weight as N increases. The resulting coefficients for the U-shaped distribution actually contains some negative weights in the central values, though it also closely resembles the uniform case. We can be assured that these filters will operate quite differently on similiar inputs.

TABLE I

Optimal filter coefficients for a filter length of three, for six parent distributions.

	a_1	a_2	a_3
Uniform	0.50000	0.00000	0.50000
Parabolic	0.44048	0.11905	0.44048
Triangular	0.39456	0.21088	0.39456
U-shaped	0.54000	-0.08000	0.54000
Laplacian	0.15168	0.69663	0.15168
Normal	0.33333	0.33333	0.33333

TABLE II

Optimal filter coefficients for a filter length of five, for six parent distributions.

	a_1	a_2	a_3	a_4	a_5
Uniform	0.50000	0.00000	0.00000	0.00000	0.50000
Parabolic	0.38629	0.07954	0.06835	0.07954	0.38629
Triangular	0.30608	0.11885	0.15014	0.11885	0.30608
U-shaped	0.55848	-0.04486	-0.02724	-0.04486	0.55848
Laplacian	0.03944	0.20322	0.51468	0.20322	0.03944
Normal	0.20000	0.20000	0.20000	0.20000	0.20000

TABLE III

Optimal filter coefficients for a filter length of seven, for six parent distributions.

	a_1	a_2	a_3	a_4
Uniform	0.50000	0.00000	0.00000	0.00000
Parabolic	0.35844	0.06736	0.05083	0.04673
Triangular	0.27444	0.06885	0.09683	0.11977
U-shaped	0.56031	-0.03607	-0.01746	-0.01356
Laplacian	0.02239	0.03344	0.23314	0.42208
Normal	0.14286	0.14286	0.14286	0.14286
	a_7	a_6	a_5	a_4

TABLE IV

Optimal filter coefficients for a filter length of nine, for six parent distributions.

	a_1	a_2	a_3	a_4	a_5
Uniform	0.50000	0.00000	0.00000	0.00000	0.00000
Parabolic	0.34031	0.06134	0.04428	0.03623	0.03568
Triangular	0.25913	0.04914	0.05734	0.08392	0.10094
U-shaped	0.55478	-0.02627	-0.01460	-0.00988	-0.00806
Laplacian	-0.01899	0.02904	0.06965	0.23795	0.36469
Normal	0.11111	0.11111	0.11111	0.11111	0.11111
	a_9	a_8	a_7	a_6	a_5

TABLE V

Optimal filter coefficients for a filter length of eleven, for six parent distributions.

	α_1	α_2	α_3	α_4	α_5
Uniform	0.50000	0.00000	0.00000	0.00000	0.00000
Parabolic	0.32734	0.05758	0.04008	0.03234	0.02879
Triangular	0.24886	0.04358	0.03777	0.05078	0.07495
U-shaped	0.54344	-0.01349	-0.01230	-0.00847	-0.00633
Laplacian	0.01320	-0.00178	0.01390	0.08628	0.22814
Normal	0.09091	0.09091	0.09091	0.09091	0.09091
	α_{11}	α_{10}	α_9	α_8	α_7

	α_6
Uniform	0.00000
Parabolic	0.02774
Triangular	0.08813
U-shaped	-0.00570
Laplacian	0.32054
Normal	0.09091
	α_6

TABLE VI

Optimal filter coefficients for a filter length of thirteen, for six parent distributions.

	a_1	a_2	a_3	a_4	a_5
Uniform	0.50000	0.00000	0.00000	0.00000	0.00000
Parabolic	0.31707	0.05520	0.03741	0.02925	0.02700
Triangular	0.24114	0.04165	0.03021	0.03127	0.04825
U-shaped	0.52902	-0.00028	-0.00897	-0.00719	-0.00593
Laplacian	0.00038	0.00013	0.00498	0.02869	0.09913
Normal	0.07692	0.07692	0.07692	0.07692	0.07692
	a_{13}	a_{12}	a_{11}	a_{10}	a_9

	a_6	a_7
Uniform	0.00000	0.00000
Parabolic	0.02230	0.02355
Triangular	0.06856	0.07785
U-shaped	-0.00443	-0.00445
Laplacian	0.22111	0.29117
Normal	0.07692	0.07692
	a_8	a_7

TABLE VII

Optimal filter coefficients for a filter length of fifteen, for six parent distributions.

	a_1	a_2	a_3	a_4	a_5
Uniform	0.50000	0.00000	0.00000	0.00000	0.00000
Parabolic	0.30906	0.05306	0.03599	0.02795	0.02353
Triangular	0.23559	0.03944	0.02730	0.02441	0.03005
U-shaped	0.51584	0.00715	-0.00228	-0.00591	-0.00531
Laplacian	0.00949	-0.00001	-0.00374	0.00813	0.03673
Normal	0.06667	0.06667	0.06667	0.06667	0.06667
	a_{15}	a_{14}	a_{13}	a_{12}	a_{11}

	a_6	a_7	a_8
Uniform	0.00000	0.00000	0.00000
Parabolic	0.02103	0.01973	0.01931
Triangular	0.04476	0.06279	0.07132
U-shaped	-0.00423	-0.00357	-0.00336
Laplacian	0.10682	0.20989	0.26537
Normal	0.06667	0.06667	0.06667
	a_{10}	a_9	a_8

TABLE VIII

Optimal filter coefficients for a filter length of seventeen, for six parent distributions.

	α_1	α_2	α_3	α_4	α_5
Uniform	0.50000	0.00000	0.00000	0.00000	0.00000
Parabolic	0.30222	0.05155	0.03474	0.02669	0.02218
Triangular	0.23068	0.03850	0.02599	0.02095	0.02140
U-shaped	0.50696	0.00851	0.00349	-0.00290	-0.00452
Laplacian	0.00831	0.00106	-0.00468	0.00196	0.01272
Normal	0.05882	0.05882	0.05882	0.05882	0.05882
	α_{17}	α_{16}	α_{15}	α_{14}	α_{13}

	α_6	α_7	α_8	α_9
Uniform	0.00000	0.00000	0.00000	0.00000
Parabolic	0.01947	0.01784	0.01697	0.01669
Triangular	0.02862	0.04273	0.05839	0.06549
U-shaped	-0.00399	-0.00331	-0.00288	-0.00274
Laplacian	0.04448	0.11210	0.20104	0.24603
Normal	0.05882	0.05882	0.05882	0.05882
	α_{12}	α_{11}	α_{10}	α_9

TABLE IX

Optimal filter coefficients for a filter length of nineteen, for six parent distributions.

	a_1	a_2	a_3	a_4	a_5
Uniform	0.50000	0.00000	0.00000	0.00000	0.00000
Parabolic	0.29641	0.05032	0.03370	0.02577	0.02121
Triangular	0.22651	0.03778	0.02526	0.01949	0.01748
U-shaped	0.50211	0.00680	0.00572	0.00113	-0.00284
Laplacian	0.00738	0.00190	-0.00483	0.00002	0.00415
Normal	0.05263	0.05263	0.05263	0.05263	0.05263
	a_{19}	a_{18}	a_{17}	a_{16}	a_{15}

	a_6	a_7	a_8	a_9	a_{10}
Uniform	0.00000	0.00000	0.00000	0.00000	0.00000
Parabolic	0.01838	0.01657	0.01548	0.01484	0.01465
Triangular	0.01975	0.02769	0.04099	0.05471	0.06071
U-shaped	-0.00358	-0.00315	-0.00267	-0.00238	-0.00229
Laplacian	0.01707	0.05107	0.11538	0.19280	0.23011
	a_{14}	a_{13}	a_{12}	a_{11}	a_{10}

TABLE X

Optimal filter coefficients for a filter length of twenty-one, for six parent distributions.

	α_1	α_2	α_3	α_4	α_5
Uniform	0.50000	0.00000	0.00000	0.00000	0.00000
Parabolic	0.29137	0.04922	0.03292	0.02506	0.02040
Triangular	0.22290	0.03714	0.02479	0.01881	0.01572
U-shaped	0.50025	0.00344	0.00574	0.00425	-0.00045
Laplacian	0.00663	0.00254	-0.00473	-0.00062	0.00122
Normal	0.04762	0.04762	0.04762	0.04762	0.04762
	α_{21}	α_{20}	α_{19}	α_{18}	α_{17}

	α_6	α_7	α_8	α_9	α_{10}
Uniform	0.00000	0.00000	0.00000	0.00000	0.00000
Parabolic	0.01761	0.01568	0.01442	0.01362	0.01318
Triangular	0.01544	0.01879	0.02704	0.03945	0.05156
U-shaped	-0.00258	-0.00289	-0.00256	-0.00222	-0.00201
Laplacian	0.00621	0.02130	0.05660	0.11725	0.18524
Normal	0.04762	0.04762	0.04762	0.04762	0.04762
	α_{16}	α_{15}	α_{14}	α_{13}	α_{12}

	α_{11}
Uniform	0.00000
Parabolic	0.01303
Triangular	0.05671
U-shaped	-0.00194
Laplacian	0.21673
Normal	0.04762
	α_{11}

TABLE XI

Optimal filter coefficients for a filter length of twenty-three, for six parent distributions.

	α_1	α_2	α_3	α_4	α_5
Uniform	0.50000	0.00000	0.00000	0.00000	0.00000
Parabolic	0.28692	0.04837	0.03224	0.02437	0.01985
Triangular	0.21970	0.03663	0.02442	0.01834	0.01497
U-shaped	0.49845	0.00283	0.00422	0.00422	0.00240
Laplacian	0.00602	0.00301	-0.00453	-0.00087	0.00024
Normal	0.04348	0.04348	0.04348	0.04348	0.04348
	α_{23}	α_{22}	α_{21}	α_{20}	α_{19}

	a_6	a_7	a_8	a_9	a_{10}
Uniform	0.00000	0.00000	0.00000	0.00000	0.00000
Parabolic	0.01702	0.01495	0.01369	0.01276	0.01215
Triangular	0.01347	0.01411	0.01825	0.02653	0.03807
U-shaped	-0.00083	-0.00239	-0.00237	-0.00212	-0.00188
Laplacian	0.00213	0.00838	0.02532	0.06120	0.11813
Normal	0.04348	0.04348	0.04348	0.04348	0.04348
	a_{18}	a_{17}	a_{16}	a_{15}	a_{14}

	a_{11}	a_{12}
Uniform	0.00000	0.00000
Parabolic	0.01182	0.01172
Triangular	0.04883	0.05333
U-shaped	-0.00172	-0.00167
Laplacian	0.17832	0.20529
Normal	0.04348	0.04348
	a_{13}	a_{12}

TABLE XII

Optimal filter coefficients for a filter length of twenty-five, for six parent distributions.

	α_1	α_2	α_3	α_4	α_5
Uniform	0.50000	0.00000	0.00000	0.00000	0.00000
Parabolic	0.28292	0.04758	0.03169	0.02393	0.01936
Triangular	0.21685	0.03614	0.02411	0.01811	0.01457
U-shaped	0.49732	0.00177	0.00320	0.00387	0.00394
Laplacian	0.00550	0.00335	-0.00427	-0.00101	-0.00008
Normal	0.04000	0.04000	0.04000	0.04000	0.04000
	α_{25}	α_{24}	α_{23}	α_{22}	α_{21}

	α_6	α_7	α_8	α_9	α_{10}
Uniform	0.00000	0.00000	0.00000	0.00000	0.00000
Parabolic	0.01641	0.01447	0.01316	0.01204	0.01147
Triangular	0.01248	0.01190	0.01341	0.01786	0.02610
U-shaped	0.00081	-0.00122	-0.00197	-0.00206	-0.00180
Laplacian	0.00065	0.00314	0.01064	0.02907	0.06499
Normal	0.04000	0.04000	0.04000	0.04000	0.04000
	α_{20}	α_{19}	α_{18}	α_{17}	α_{16}

	α_{11}	α_{12}	α_{13}
Uniform	0.00000	0.00000	0.00000
Parabolic	0.01096	0.01068	0.01066
Triangular	0.03684	0.04643	0.05039
U-shaped	-0.00162	-0.00150	-0.00146
Laplacian	0.11835	0.17195	0.19541
Normal	0.04000	0.04000	0.04000
	α_{15}	α_{14}	α_{13}

III. A CLASS OF HEAVY-TAILED NOISE DISTRIBUTIONS

Many authors have noted that the median filter is effective for suppressing impulsive type input structures, which is an extreme case of heavy-tailed noise. In this section we examine a particular class of distribution functions, dependent upon a parameter β. For certain values of β, we obtain distribution functions that can be made arbitrarily heavy-tailed, or conversely, shallow-tailed. We will concern ourselves, however, only with those distributions in the class that are at least as heavy-tailed as the Laplacian distribution, which is a member of the class. We will then apply the same techniques as used in section II to obtain optimal filter coefficients, using these distribution functions as parent functions. These coefficients will hopefully yield some interesting information concerning the form of the coefficients as the parent distributions grow more heavy-tailed.

Consider a parent density function of the form

$$f_x(x) = f_x(x;\beta) = ke^{-\gamma|x|^{\beta}} \quad ; |x| < \infty \qquad (33)$$

where γ and β are positive real numbers, and where k is chosen such that

$$\int_{-\infty}^{+\infty} f_x(x)\, dx = 1 \tag{34}$$

where integration yields

$$k = (\beta\gamma^{1/\beta})/2\Gamma(1/\beta) \tag{35}$$

with

$$\gamma = [\Gamma(3/\beta)/\Gamma(1/\beta)]^{\beta/2}\sigma_x^{-\beta} \tag{36}$$

where σ_x is the standard deviation and Γ is the ordinary gamma function.

We will use two methods to measure the weight in the tails of f_x as we vary β. Plotting f_x versus x as we vary β proves fruitless, as the asymptotic properties are too extreme to ascertain anything from an ordinary plot. Hence, in Figure 4, we plot $\log_{10}f_x(x)$ versus x as β takes on several values, with $\sigma_x = 1$. The values chosen were β = 1/4, 1/2, 3/4, 1, 3/2, 2, and 10, which gives a good representation of the behavior of f_x for β values both less than and greater than one. The Laplacian distribution is defined by $\beta = 1$, and we have, of course, a linear plot in this case. We see that as β increases beyond one, the plot of $\log_{10}f_x$ falls off very rapidly; in fact for $\beta = 10$, the plot has developed a very pronounced "knee". It is apparent

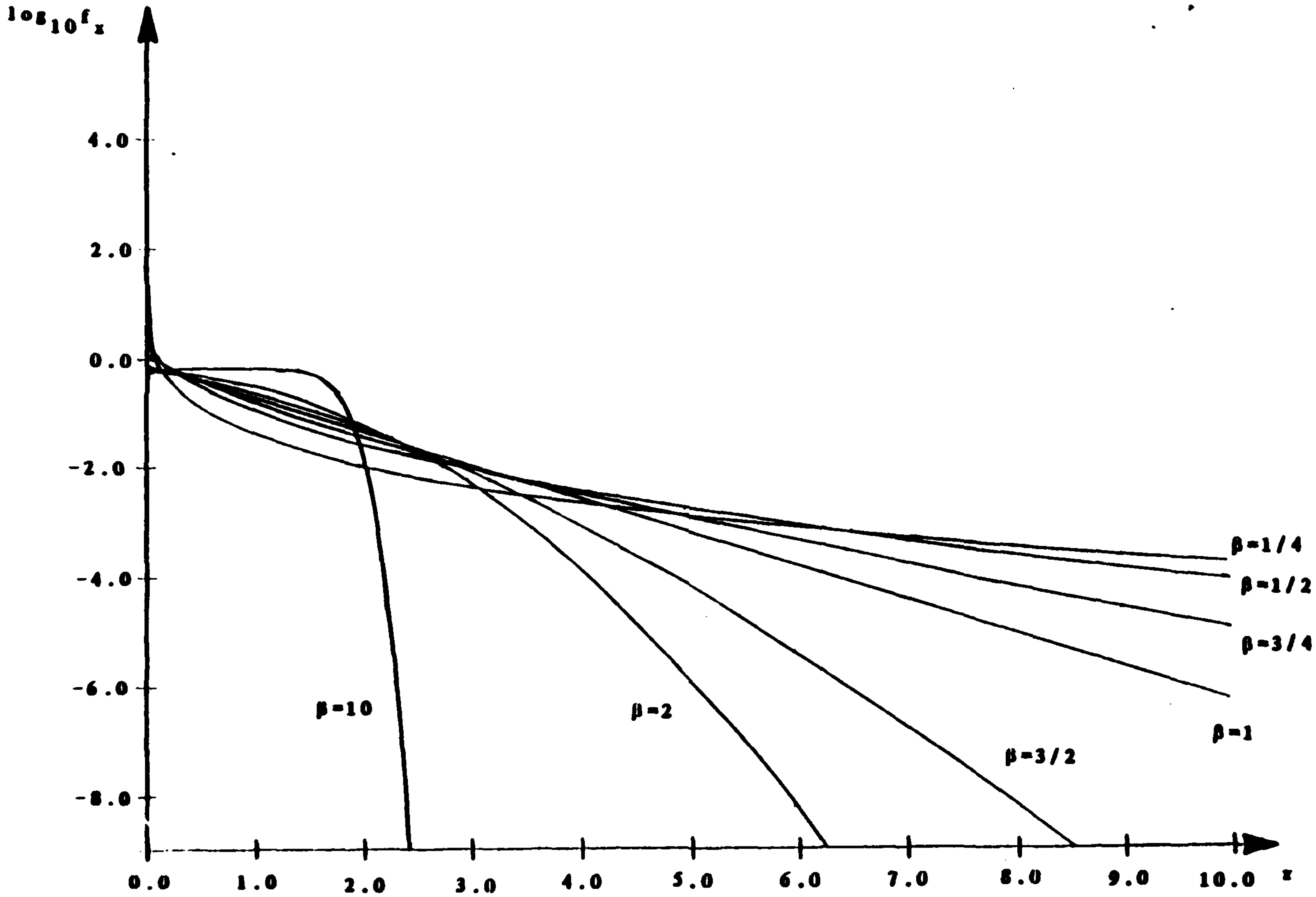


Figure 4. Plot of $\log_{10} f_x$ versus x for several values of the parameter, β

that for β values increasing beyond one, we obtain progressively shallower tails in f_x. For β values in the interval (0,1], we observe apparently increasingly heavier weight in the tails of f_x as β approaches zero, although the trend is not as pronounced as that observed for $\beta > 1$.

We would like to gain a greater insight on the behavior of f_x for β in the interval (0,1]. Consider the following criteria: denote

$$P_c(\beta) = \mathrm{Prob}\{|x|>c\} = 1 - \int_{-c}^{c} f_x(x)\, dx \tag{37}$$

where $c > 0$. The behavior of P_c as c takes on different values should give an excellent indication of the impulsivity of the additive noise distribution as a function of β, provided that we choose suitable values for c.

In Figure 5 we have plots of $P_c(\beta)$ versus β for $c = \sigma_x$, $3\sigma_x$. For β values in the interval (0,0.1] the plots are essentialy identical; it is apparent that for these values of β, f_x is very heavy-tailed, becoming remarkably so as $\beta \downarrow 0$. The plots are different for $\beta > 0.1$, in good agreement with the "knee" observed in Figure 4. The plot for $c = 3\sigma_x$ shows that f_x is considerably shallower-tailed in the region $0 < \beta < 1$, but heavier than the Laplacian ($\beta = 1$) for all β satisfying $0 < \beta < 1$.

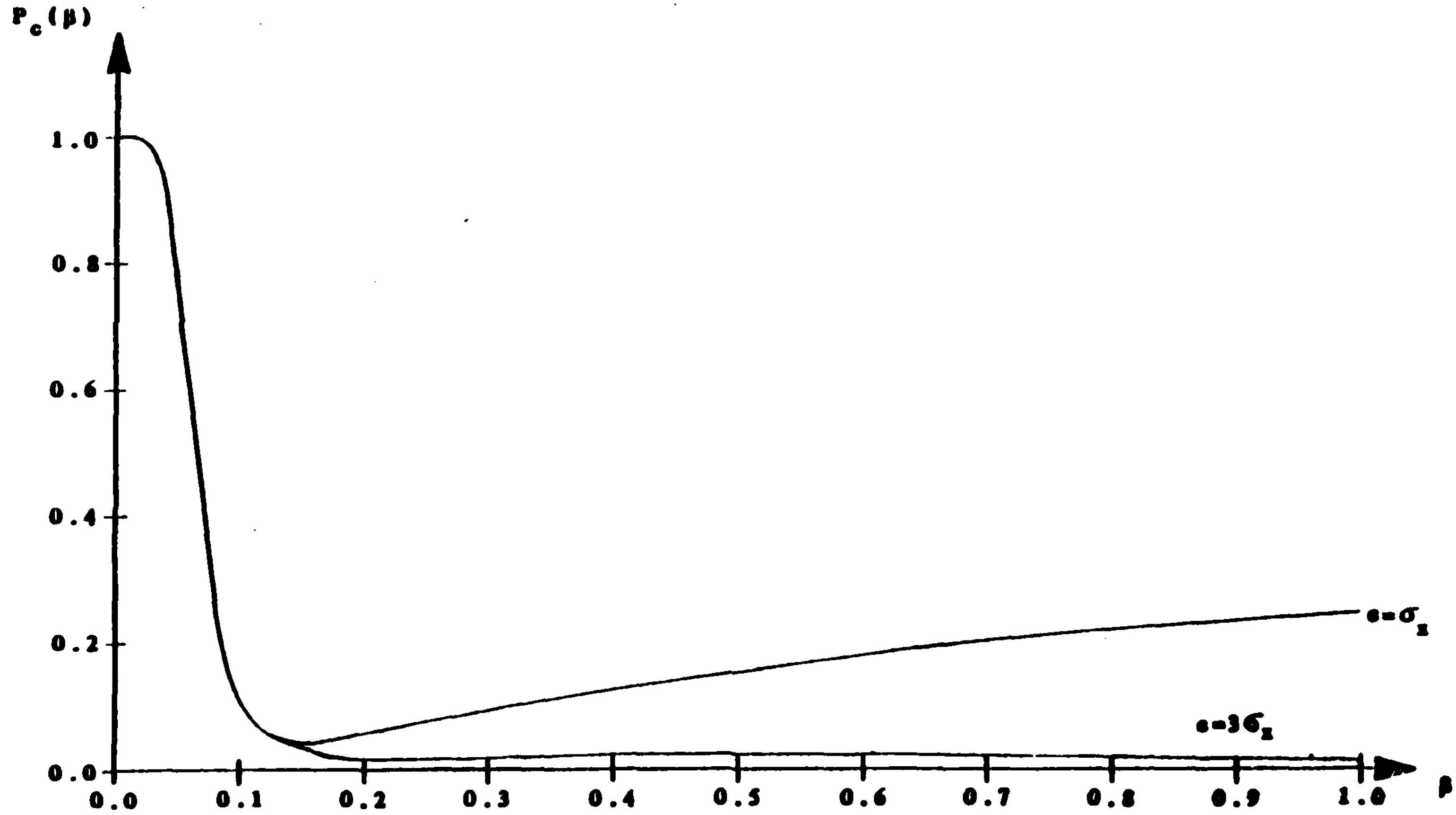


Figure 5. Plot of $P_c(\beta)$ versus β

We can now apply the techniques used in section II to obtain the optimal coefficients for various β in developing the subclass of filters associated with f_x. Ideally, it would be desirable to do this for $0 < \beta < 0.1$, as well as for other values in (0,1], but this was not done for several reasons. First, it should be noted that as β approaches zero, the constant k in (35) increases dramatically, producing a large "spike" at the origin. Because of this, the density and joint density of the order statistics in (16) and (17) are very ill-behaved. Evaluation of the elements of the covariance matrix H thereby becomes very costly, as the numerical integration requires a significant number of subintervals to attain any usable accuracy. Adaptive integration routines failed completely for small β values. Further, the distribution function F_x associated with f_x must be evaluated pointwise by numerical integration as well, for each point in the in the outer integrations in (18), yielding the problem intractable for very small β values.

These limitations are not particularly disturbing, however, since we have seen that the median filter is invariant to such extremely impulsive noise, which may be modeled by f_x for small β. We are more directly concerned with the behavior of $\underline{a}$ as β decreases past one.

The values ultimately used for β were 1/2 and 3/4, and we already have results for the Laplacian case, $\beta = 1$. The window length used was N = 3; larger window lengths would also have entailed prohibitive amounts of computation time. In Table XIII below are shown the resulting optimal coefficients for the three cases:

TABLE XIII

Optimal filter coefficients for a filter length of three, for three parent distributions $f_x(x;\beta)$, for $\beta = 1, 3/4, 1/2$.

	α_1	α_2	α_3
$\beta = 1$ (Laplacian)	0.1517	0.6966	0.1517
$\beta = 3/4$	0.0715	0.8571	0.0715
$\beta = 1/2$	0.0052	0.9897	0.0052

We see that as β grows smaller, there is a dramatic tendency towards the median; the value of α_2 is approaching one very rapidly, and the value of $\alpha_1 = \alpha_3$ is approaching zero. As we mentioned in the Introduction, this certainly strengthens the notion that the median and almost-median type filters are extremely effective for suppressing additive noise more impulsive than the Laplacian case, at least for a constant background signal. Were we to use a

criterion such as Justusson's with these distributions for edge-type input structures, the tendency toward the median would likely be even more marked. We will examine the effect that some of the filters that we designed in section II have on some simple inputs in the next section.

IV. DETERMINISTIC FILTER PROPERTIES

A. A Qualitative Study of Designed Filters

We recall that the median filter has been noted to have the often desirable properties of invariance to edges of sufficiently long duration and insensitivity to impulses of sufficiently short duration, where the sufficiency depends upon the length of the filter. We may wonder how the filters we have designed earlier treat these types of input structures. They have not been designed with these inputs in mind, or with any input structure in mind, other than the constant signal value that we chose as a simplifying assumption. Neither is there anything known yet about the deterministic behavior of these filters. There is, however, a result due to Nodes and Gallagher [7] that seems to give at least an intuitive approach to the problem.

Property: A rising impulse like signal of length less than N-i+1 points or a falling impulse-like signal of length less than i points will be eliminated by an ith ranked-order operation.

We can easily see that a single pass of a general order statistical filter is comprised of the weighted sum of the outputs of single passes of ith ranked-order operations for i = 1,...,N. There is no intuitive reason why the property

should hold for the general OSF; it seems unlikely.

Therefore it seems best to instead apply some inputs and see what happens; we can make few predictions except when the inputs are relatively simple. Figure 6a shows a simple input sequence of length 100 consisting of a single impulse of length one, another of length three, and a cluster of closely associated impulses in the center. Each impulse is of amplitude five. Figures 6b through 6f show the sequence after passes by several of the designed filters.

We can see that each of the five estimators considered here spread the impulses somewhat. Varying degrees of attenuation were observed, but the trend was very similiar for the length three uniform, parabolic, triangular, and U-shaped estimators; the single impulse was approximately halved while the center cluster was attenuated similiarly, but also squared off. The last impulse of length three was changed less, but was spread some. Interestingly, the U-shaped estimator produced a dip in the cluster of impulses in the center, and jagged edges in the outer impulse of length three.

This is in contrast to the Laplacian estimator, which like the median filter, greatly attenuated the single length-one impulse, while essentially retaining the other groups, although there was some rounding at the edges. As we can see from the property, the median filter would have

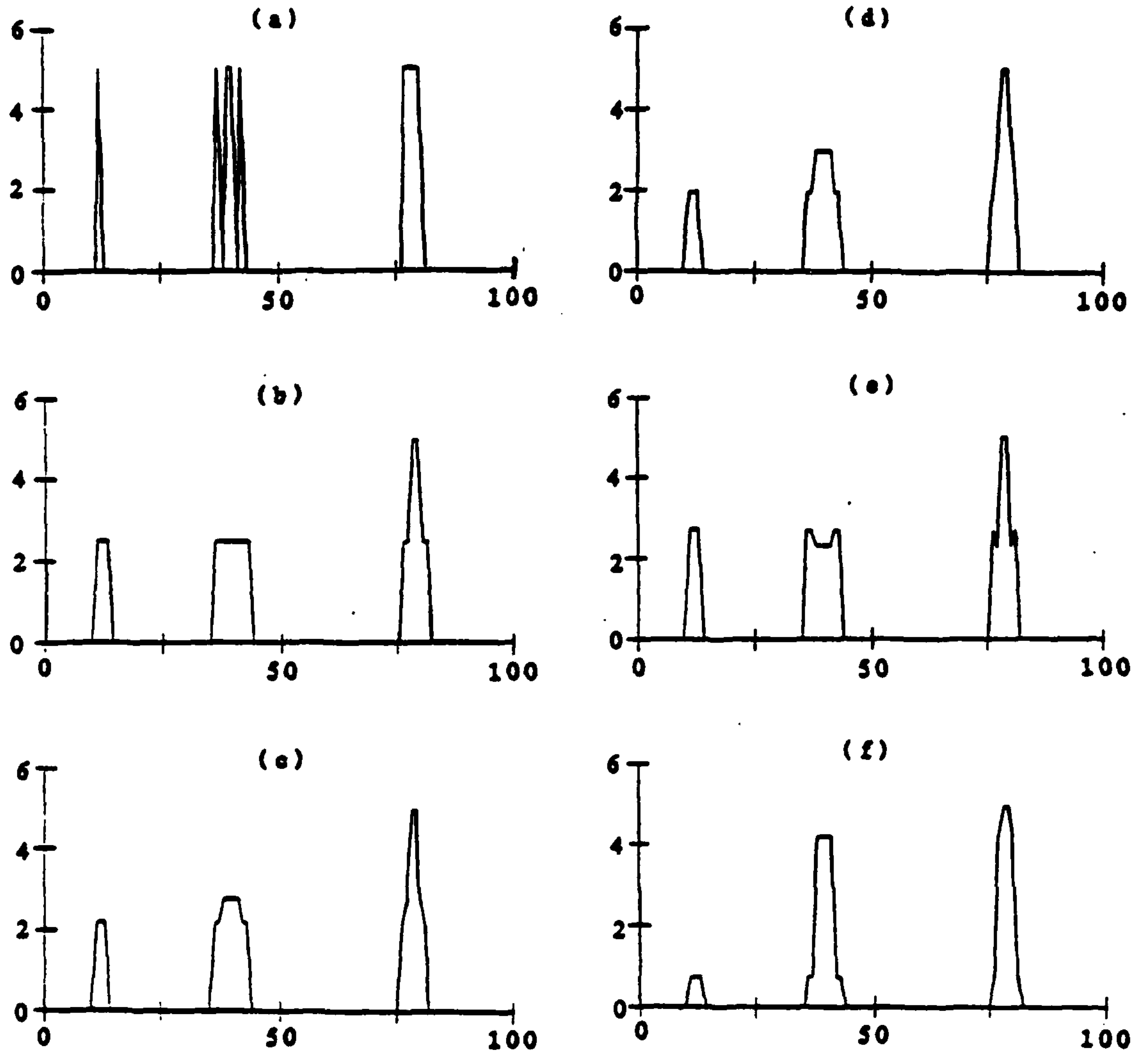


Figure 6. Three different impulse groups
(a) Unfiltered
(b) After one pass with uniform estimator, N=3
(c) After one pass with parabolic estimator, N=3
(d) After one pass with triangular estimator, N=3
(e) After one pass with U-shaped estimator, N=3
(f) After one pass with Laplacian estimator, N=3

removed the first impulse while leaving the last group unchanged. The center cluster of impulses would be reduced to an impulse of length four.

Figures 7a-7f show filtering of the same input sequence using the same classes of estimators, but of length thirteen, plus an averaging filter (normal estimator) of the same length. We can see the trend more clearly for a thirteen point window. The uniform, parabolic, triangular, and U-shaped estimators smeared the impulse groups much more, but with no significant increase in attenuation. The normal estimator performed as expected, both attenuating and smearing. The Laplacian estimator nearly eliminated the lone impulse, and greatly attenuated the other groups, whereas the median filter of the same length would have eliminated all three groups.

We can see from these examples that the estimators with most of the weight in the outer α_i are intuitively and apparently ineffective for removal of impulsive noise, which is supported by the results of the last section. In fact, those filters may be more useful for detecting impulses, rather than eliminating them, by enlarging them. There could be application in growing small regions in images.

Let us now examine how these filters treat a simple edge uncorrupted by any noise. Figure 8a shows the edge that we shall consider. Figures 8b-8d show the edge after single passes by length three uniform, U-shaped, and

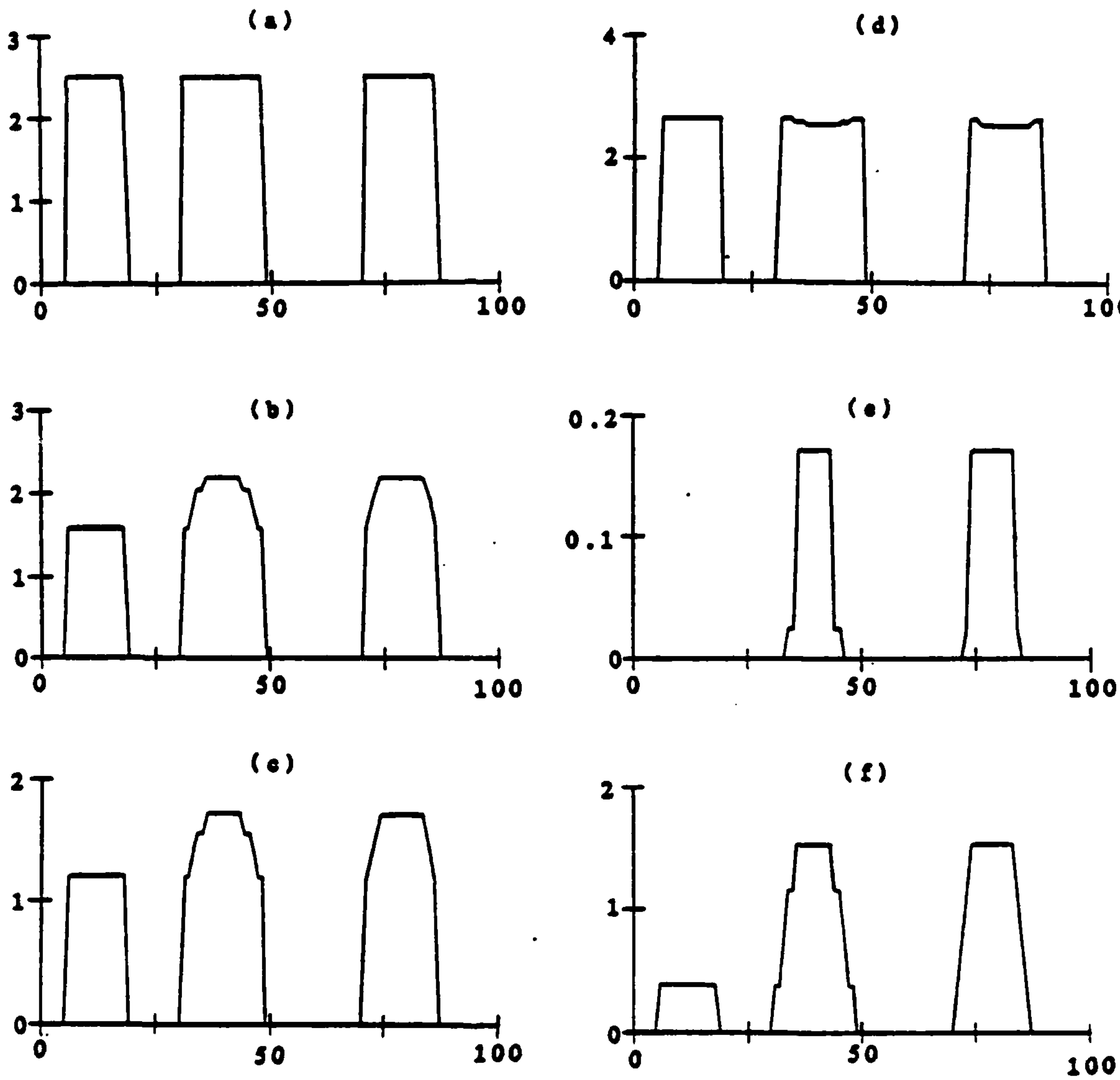


Figure 7. Same impulse groups as Figure 7
(a) After one pass with uniform estimator, N=13
(b) After one pass with parabolic estimator, N=13
(c) After one pass with triangular estimator, N=13
(d) After one pass with U-shaped estimator, N=13
(e) After one pass with Laplacian estimator, N=13
(f) After one pass with Normal estimator, N=13

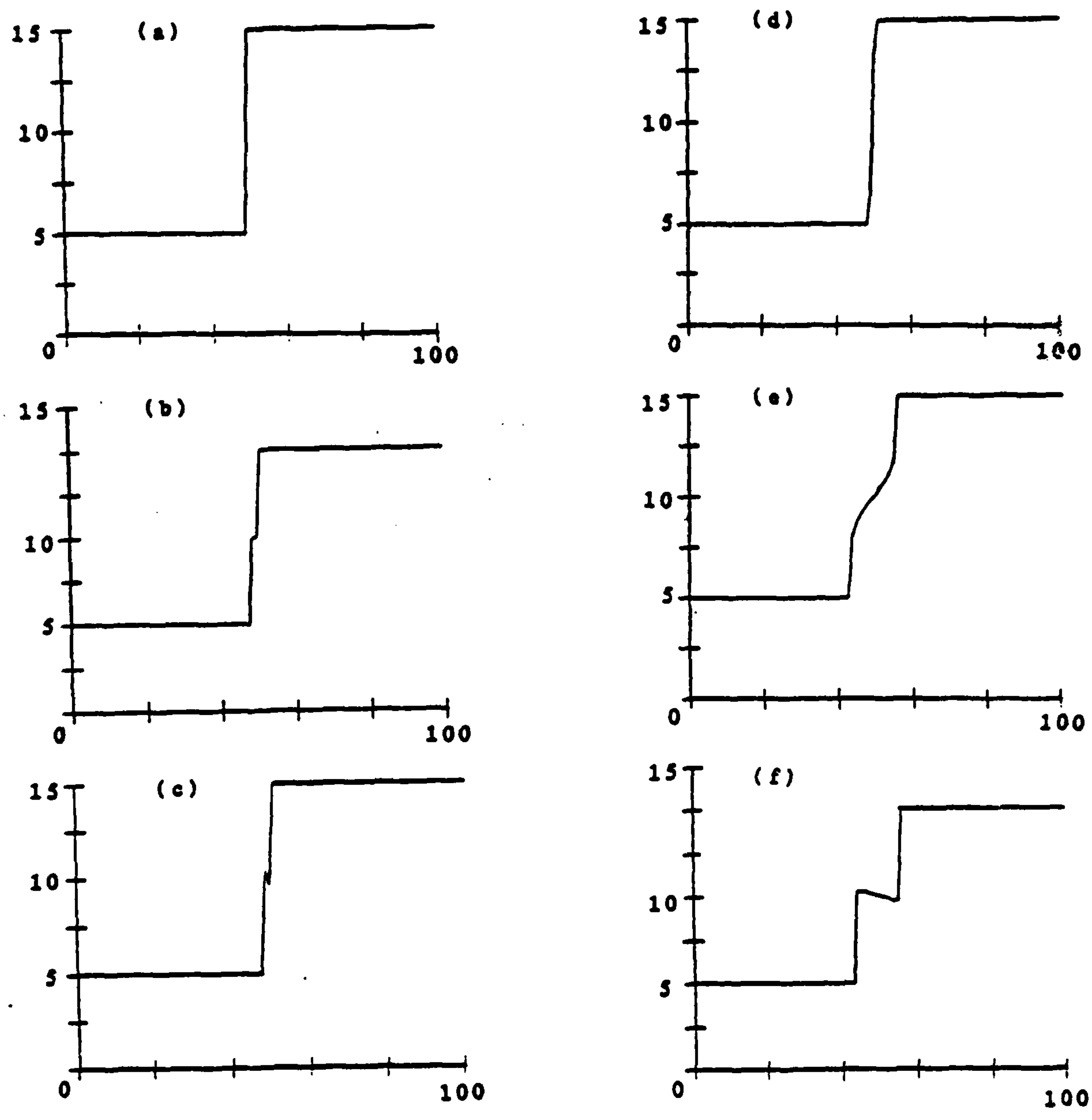


Figure 8. Simple edge with height five
(a) Unfiltered
(b) After one pass with uniform estimator, N=3
(c) After one pass with U-shaped estimator, N=3
(d) After one pass with Laplacian estimator, N=3
(e) After one pass with parabolic estimator, N=13
(f) After one pass with U-shaped estimator, N=13

Laplacian estimators, respectively. We see that the uniform and U-shaped estimators produced disturbances along the edge, which may be highly undesireable. The Laplacian estimator, however, preserved the edge rather well, as a running median would do. We can see a general trend here in that OSFs with most of the weight spread among the central coefficients perform like the median with regard to edges and impulses, and filters with most of the weight situated among the outer coefficients perform in an opposite sense. This is shown quite clearly in Figures 8e and 8f which show passes by the parabolic and U-shaped thirteen point filter windows. The edge is severely corrupted.

Let us examine momentarily a more complicated input sequence, shown in Figure 9a. This sequence contains various impulses, edges, and ramps, as well as constant regions. Figures 9b-9f show the input after filtering by uniform, parabolic, U-shaped, Laplacian, and median filtering, with a filter length of *five* in all cases. We see that the uniform and parabolic filters kept the general structure intact, but increased the width of the impulses, while approximately halving their magnitude. In an image, small light or dark areas would become more perceptible after this kind of filtering. Note that level areas were unaffected, due to the constraint given in (12). The U-shaped filter produced many spurious points and edges after a single pass, though behaving similiarly to the uniform estimator otherwise. The Laplacian and median

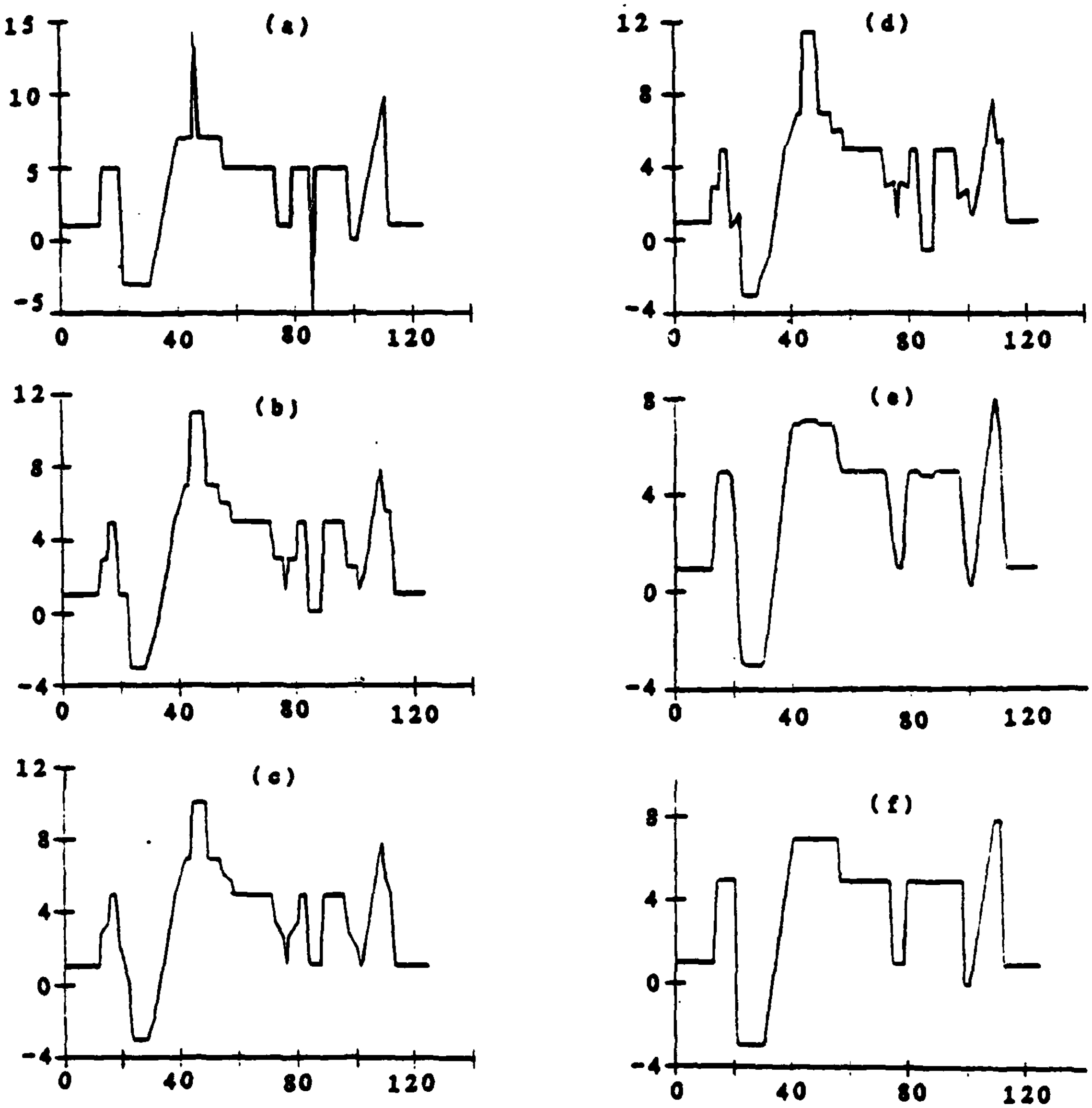


Figure 9. More complicated input signal
(a) Unfiltered
(b) After one pass with uniform estimator, N=5
(c) After one pass with parabolic estimator, N=5
(d) After one pass with U-shaped estimator, N=5
(e) After one pass with Laplacian estimator ,N=5
(f) After one pass with median filter, N=5

filters performed as expected, preserving the picture almost exactly except for removing the impulse, and in the case of the Laplacian filter, gently rounding the edges.

B. Synthesis of Root Sets

A known property of median filters is the existence of a root set for a given filter length. Aarce and Gallagher [18] have found a method for generating a set of binary roots for a median filter of a given window length, culminating in an analytic expression for the number of binary roots for a median filter of any (odd) window length.

Here we attempt a similiar examination of the class of root sets for a given filter $\underline{\alpha}$. The analysis is much more difficult than for the median binary case; in fact we do not consider finite length signals since the problem of terminating a synthesized root signal is not yet resolved.

Consider an N-point (N odd) arbitrary OSF with coefficients $\alpha_1,\ldots,\alpha_N$ where the sum of the coefficients is one, to insure unbiasedness. We need not specify that $\alpha_i = \alpha_{N-i+1}$ (symmetry) for this analysis, though they are the subclass of filters we are most interested in. Let us denote $\{x_i\ ;\ i = 1,\ldots,N\}$ as the set of input samples currently contained within the filter window, and $\{x_{(i)}\ ;\ i = 1,\ldots,N\}$ as their order statistics. Let us further denote y_i, $i = 1,\ldots.$ as the input root set that we

will synthesize. Note that when the filter is operating upon samples $y_{i-M},\ldots,y_i,\ldots,y_{i+M}$ where $M = (N-1)/2$, that they are considered as $x_1,\ldots,x_N$.

A fixed point or root element can be seen to be defined by the equation

$$\sum_{i=1}^{N} a_i x_{(i)} = x_{(N+1)/2} \tag{38}$$

Note that the quantity on the right side of (38) is not an order statistic.Suppose now that we are given any real number ξ that we would like to be the first element of the root sequence, that is $y_1 = \xi$. In accordance with standard median filtering practice we will append $(N+1)/2$ points with value ξ to the left of y_1. We will see now that our filter coefficients and our selection of the first input value constrains the values of the next $(N-1)/2$ points in the input sequence, $y_2,\ldots,y_{(N+1)/2}$. What we must first consider is the ordering of the respective values of y_i, $i = 1,\ldots,N$, as this will define the regions in $(N-1)/2$-space that give a fixed-point for the first input value.

Let us define Q as the set of subscript values of the order statistics which correspond to $y_1 = \xi$; that is, if $x_{(p)}=x_{(p+1)}=\ldots=x_{(p+(N-1)/2)}=y_1=\xi$, then we define $Q = \{p,\ldots,p+(N-1)/2\}$, and $Q^c = \{1,\ldots,p-1,p+(N+1)/2,\ldots,N\}$. ($Q^c$ = Q complement)

Then, for the filter to be invariant to the first input value, we have the following relation:

$$\sum_{i=1,\, i\in Q^c}^{N} \alpha_i x_{(i)} + \sum_{i=1,\, i\in Q}^{N} \alpha_i y_1 = y_1 \qquad (39)$$

or

$$\sum_{i=1,\, i\in Q^c}^{N} \alpha_i x_{(i)} = y_1\left[1 - \sum_{i=1,\, i\in Q}^{N} \alpha_i\right] \qquad (40)$$

or

$$\sum_{i=1,\, i\in Q^c}^{N} \alpha_i x_{(i)} = y_1 \sum_{i=1,\, i\in Q^c}^{N} \alpha_i \qquad (41)$$

since

$$\sum_{i=1,\, i\in Q}^{N} \alpha_i + \sum_{i=1,\, i\in Q^c}^{N} \alpha_i = \sum_{i=1}^{N} \alpha_i = 1 \qquad (42)$$

Let us then note that although the OSF is nonlinear, it has the following property: given an input sequence $\{z_n\}$, $OSF[\{z_n+\xi\}] = OSF[\{z_n\}] + \xi$, where ξ is any real constant. Hence, we may define our first input point to be zero in constructing our root sequence since the signal is merely changed by a constant value throughout. Hence, defining $y_1 = 0$, we obtain from (41):

$$\sum_{i=1,\, i\in Q^c}^{N} \alpha_i x_{(i)} = 0 \qquad (43)$$

Hence, in synthesizing a root signal with first value given by $y_1 = \xi$, we proceed using (43) and then use $\{y_i+\xi\} \doteq \{y'_i\}$ as our desired root signal.

There are $((N+1)/2)!$ possible permutations of the ordering of the y_i for $i = 1,\ldots,(N+1)/2$, but only some of these will be reconcilable with (43). Let us consider an example.

<u>Example 1:</u> Consider a filter of length five with coefficients given by $\underline{a}^T = (1/4,-1/4,1,-1/4,1/4)$. We then have three choices for Q: $Q_1 = \{1,2,3\}$, $Q_2 = \{2,3,4\}$, and $Q_3 = \{3,4,5\}$. We will consider eac' case separately:

<u>a</u>. Let $Q = Q_1$. The solution for (y_2,y_3) resulting in a root at y_1 is given by

$$\text{(i)}\quad (y_2,y_3) = \{(a,a)\colon a \geq 0\}$$

<u>b</u>. Let $Q = Q_2$. The solution here is

$$\text{(ii)}\quad (y_2,y_3) = \{(a,-a)\colon a \text{ real}\}$$

<u>c</u>. Let $Q = Q_3$. Here we have

$$\text{(iii)}\quad (y_2,y_3) = \{(a,a)\colon a \leq 0\}$$

Solutions (i)-(iii) define a complete solution for (y_2, y_3) given by $(y_2, y_3) = \{(a,b): |a| = |b|\}$, such that the the filter will be invariant to y_1. We have a symmetric filter in this case, resulting in a symmetric solution set.

The following examples consider other filters of length five; only the complete solutions are given, since they are simple to arrive at.

<u>Example 2:</u> Let $\underline{a}^T = (1/2,0,0,0,1/2)$, the optimal estimator for the uniform distribution for a constant background. The solution here is given by $(y_2, y_3) = \{(a,-a): a \text{ real}\}$

<u>Example 3:</u> Let $\underline{a}^T = (-1/2,3/2,-1,3/2,-1/2)$. The solution for this symmetrical filter is given by $(y_2, y_3) = \{(a,-a): a \text{ real}\} \cup \{(a,a/3): a \text{ real}\} \cup \{(a,3a): a \text{ real}\}$.

<u>Example 4:</u> Let $\underline{a}^T = (1,-1/2,1,1/2,-1)$, an antisymmetrical filter. The only solution in the case is $(y_2, y_3) = (0,0)$, which is a trivial solution for any OSF.

<u>Example 5:</u> Let $\underline{a}^T = (1/4,1,-1/2,-1/4,1/2)$, a very nonsymmetric filter. The solution here is given by $(y_2, y_3) = \{(a,-a/2): a \leqslant 0\} \cup \{(a,-2a): a \geqslant 0\}$ which is nonsymmetric as well.

It is clear that the solutions vary for different filters, and some may have only the *origin* as a solution. In this case, it is simple to see that the root signal must be $y_i = 0$ for all i, since we are confronted with an identical situation for each $i > 1$.

Thus far we have described a general method for finding the set of values $\{y_2,\ldots,y_{(N+1)/2}\}$ that will give a fixed point at y_1. The next step will be to examine the process of extracting the remainder of the root signal. This must be done by a point-by-point process. Thus, assuming we have selected values $y_2,\ldots,y_{(N+1)/2}$ we next determine the value of $y_{(N+3)/2}$ so that y_2 will be a fixed point. This is equivalent to

$$\sum_{i=1}^{N} \alpha_i x_{(i)} = y_2 = x_{(N+1)/2} \tag{44}$$

For a given ordering of $\{y_1,\ldots,y_{(N+3)/2}\}$, equation (44) defines either a single point or the empty set, since $y_{(N+3)/2}$ is the only unknown. Let us again proceed by example.

<u>Example 6:</u> Consider the same filter as in Example 1. With $y_1 = 0$, the set of solutions for (y_2, y_3) was given by $\{(a,b): |a| = |b|\}$. Suppose then that we take $y_2 = -5$, $y_3 = 5$. We may then order $y_{(N+3)/2} = y_4$ relative to y_1, y_2, and y_3 in four different ways, which we will consider separately:

(i) $y_4 \leqslant -5 = y_2$

For this ordering, filtering requires that $(y_4, y_2, y_1, y_1, y_3)\underline{a} = y_2$, or $y_4 = -30$. This satisfies $y_4 \leqslant -5$, so we have a viable solution.

(ii) $-5 = y_2 \leqslant y_4 \leqslant 0 = y_1$

In this case, we require that $(y_2, y_4, y_1, y_1, y_3)\underline{a} = y_2$, or $y_4 = 20$, which does not agree with the ordering. This does not present a solution.

(iii) $y_1 = 0 \leqslant y_4 \leqslant 5 = y_3$

Here we require that $(y_2, y_1, y_1, y_4, y_3)\underline{a} = y_2$, or $y_4 = 20$, which is again irreconcilable with the ordering.

(iv) $y_3=5\leq y_4$

This entails $(y_2,y_1,y_1,y_3,y_4)\underline{a} = y_2$, or $y_4 = -10$, again presenting no solution.

For the values of (y_2,y_3) we chose, there was only one solution for y_4. It may be, however that for other choices of (y_2,y_3) there are more solutions for y_4. In fact, given $(y_1,y_2,y_3,y_4) = (0,-5,5,-30)$ there are three solutions for y_5, given by -35, -25, and 30, certainly a wide spread of values to choose from.

It is apparent that the description of the entire root set for a given filter may be quite difficult to obtain. We do however, have the following simple theorem for symmetric filters of length three:

<u>Theorem:</u> The only root signal of a non-median symmetric OSF of length three is a constant signal.

<u>Proof:</u> We have two possibilities for Q; $Q_1 = \{1,2\}$ and $Q_2 = \{2,3\}$. Considering each separately:

<u>a</u>. For $Q = Q_1$, we have $y_1=0\leq y_2$, so that $\alpha_3 y_2=0$, or $y_2=0$ unless $\alpha_3 = \alpha_1 = 0$, the median filter.

<u>b</u>. For $Q = Q_2$, we have $y_2\leq y_1=0$, which gives the same conclusion.

Since $y_2 = 0$ for the non-median case, and we are presented with the same situation for each point following, and if we recall the comments following (43), the theorem is proved.

A more complete and conclusive study of the class of root sets would prove valuable in both understanding the general deterministic properties of the class of filters we have considered, of which little is known, and also in formulating applications, such as coding.

V. CONCLUSION

In this thesis we have introduced a new class of filters, of which the median filter is a member. It naturally follows that we can do no worse than the median filter in any application that we may choose, and we may do better, if we design properly. We have used a design procedure to optimally remove various types of noise from constant signals, but more importantly we have developed a systematic method of design for optimization criteria imposed on the order statistics of the input samples, such as that suggested by Justusson for insuring the sharpness of an edge. This method of filtering may be desirable for the inherent deterministic properties, of which so little is known, or for the relative speed that can be attained, which probably is similiar to that attained for the median. We also computed the correlation matrices of the order statistics for many parent distributions, which may prove useful in later endeavors. The specific filters designed here could prove useful as well; they generally performed on a level comparable with the averaging filter in some cases and the median filter in others.

The analysis of the deterministic properties of the OSF is still a very open-ended topic; like the median filter, a greater understanding of these properties for the general filter is needed; perhaps a sinusoidal analysis could be

done. Other statistical analyses should prove valuable as well, such as consideration of non-white additive noise.

REFERENCES

[1] J.W. Tukey, Exploratory Data Analysis. Reading, Mass: Addison-Wesley, 1977.

[2] L.R. Rabiner, M.R. Sambur, and C.E. Schmidt, "Applications of a Nonlinear Smoothing Algorithm to Speech Processing," IEEE Trans. Acoust., Speech, Signal Processing, Vol. ASSP-23, pp. 552-557, 1975.

[3] N.S. Jayant, "Average and Median-Based Smoothing Techniques for Improving Digital Speech Quality in the Presence of Transmission Errors," IEEE Trans. Commun., Vol. COM-24, pp. 1043-1045, 1976.

[4] W.K. Pratt, Digital Image Processing. New York: John Wiley and Sons, 1978.

[5] T.S. Huang, G.J. Yang, and G.Y. Tang, "A Fast Two-Dimensional Median Filtering Algorithm," IEEE Trans. Acoust., Speech, Signal Processing, Vol. ASSP-27, pp. 13-18, 1979.

[6] E. Ataman, V.K. Aatre, and K.M. Wong, "A Fast Method for Real-Time Median Filtering," IEEE Trans. Acoust., Speech, Signal Processing,Vol. ASSP-28, pp. 415-420, 1980.

[7] T.M. Nodes and N.C. Gallagher., Jr. "Median Filtering," Proc. of the Allerton Conference on Communications, Control, and Computing, pp. 926-934, October, 1980.

[8] B.I. Justusson, "Median Filtering: Statistical Properties," Two-Dimensional Transforms and Filters II, T.S. Huang, ed. Berlin, Germany: Springer-Verlag, Chap. 5, 1981.

[9] H.A. David, Order Statistics. New York: John Wiley and Sons, 1981.

[10] A.E. Sarhan, "Estimation of the Mean and Standard Deviation by Order Statistics," Ann. Math. Stat., Vol. 25, pp. 317-328, 1954.

[11] A.E. Sarhan, "Estimation of the Mean and Standard Deviation by Order Statistics, Part II," Ann. Math. Stat., Vol. 26, pp. 505-511, 1955.

[12] A.E. Sarhan, "Estimation of the Mean and Standard Deviation by Order Statistics, Part III," Ann. Math. Stat., Vol. 26, pp. 576-592, 1955.

[13] D. Teichrow, "Tables of Lower Moments of Order Statistics for Samples from the Normal Distribution," Contributions to Order Statistics, A.E. Sarhan and B.G. Greenberg, eds. New York: John Wiley and Sons, 1962, Chap. 10A.

[14] M.S. Bazaraa and C.M. Shetty, Nonlinear Programming. New York: John Wiley and Sons, 1979.

[15] Contributions to Order Statistics, A.E. Sarhan and B.G. Greenberg, eds. New York:John Wiley and Sons, 1962.

[16] E.H. Lloyd, "Generalized Least-Squares Theorem," Contributions to Order Statistics, A.E. Sarhan and B.G. Greenberg, eds. New York: John Wiley and Sons, 1962, Chap. 3.

[17] S.-G. Tyan, "Median Filters: Deterministic Properties," Two-Dimensional Transforms and Filters II, T.S. Huang, ed. Berlin, Germany: Springer-Verlag, Chap. 6, 1981.

[18] G.R. Aarce and N.C. Gallagher, Jr., "State Description for the Root-Signal Set of Median Filters, to appear in IEEE Trans. Acoust., Speech, Signal Processing.

APPENDIX

The following correlation matrices in Tables XIV-XXV are symmetric across both main diagonals, and therefore only a single quadrant of each is shown. To obtain the complete matrices, one need only apply equations (19).

TABLE XIV

Correlation matrices of the order statistics of the additive noise for a filter length of three, for six parent distributions.

UNIFORM:

	1	2
1	1.20000	
2	0.30000	0.60000
3	-0.60000	

U-SHAPED:

	1	2
1	1.09091	
2	0.29221	0.81818
3	-0.58442	

PARABOLIC:

	1	2
1	1.23377	
2	0.29221	0.53247
3	-0.58442	

LAPLACIAN:

	1	2
1	2.68056	
2	0.42796	0.63889
3	-0.85593	

TRIANGULAR:

	1	2
1	0.31339	
2	0.07143	0.12321
3	-0.14286	

NORMAL:

	1	2
1	1.27566	
2	0.27566	0.44867
3	-0.55132	

TABLE XV

Correlation matrices of the order statistics of the additive noise for a filter length of five, for six parent distributions.

UNIFORM:

	1	2	3
1	1.57143		
2	0.85714	0.71429	
3	0.14286	0.28571	0.42857
4	-0.57143	-0.14286	
5	-1.28571		

U-SHAPED:

	1	2	3
1	1.25668		
2	0.81536	0.88235	
3	0.10767	0.36833	0.72193
4	-0.62333	-0.22100	
5	-1.11506		

PARABOLIC:

	1	2	3
1	1.67274		
2	0.84092	0.64671	
3	0.14801	0.25129	0.36111
4	-0.53039	-0.12021	
5	-1.29947		

LAPLACIAN:

	1	2	3
1	3.99372		
2	1.25672	0.83069	
3	0.22997	0.27950	0.35118
4	-0.67522	-0.11375	
5	-2.06820		

TRIANGULAR:

	1	2	3
1	0.43297		
2	0.20657	0.15206	
3	0.03704	0.05752	0.07995
4	-0.12557	-0.02652	
5	-0.32462		

NORMAL:

	1	2	3
1	1.80002		
2	0.80001	0.55656	
3	0.14815	0.20844	0.28683
4	-0.46992	-0.09510	
5	-1.27826		

TABLE XVI

Correlation matrices of the order statistics of the additive noise for a filter length of seven, for six parent distributions.

UNIFORM:

	1	2	3	4
1	1.83333			
2	1.25000	1.00000		
3	0.66667	0.58333	0.50000	
4	0.08333	0.16667	0.25000	0.33333
5	-0.50000	-0.25000	0.00000	
6	-1.08333	-0.66667		
7	-1.66667			

PARABOLIC:

	1	2	3	4
1	2.00188			
2	1.25236	0.93439		
3	0.64785	0.52250	0.42728	
4	0.09237	0.15229	0.20979	0.27288
5	-0.45802	-0.20966	0.00344	
6	-1.04396	-0.59152		
7	-1.74296			

TRIANGULAR:

	1	2	3	4
1	0.52477			
2	0.[illegible]196	0.22499		
3	0.15682	0.12143	0.09591	
4	0.02358	0.03569	0.04616	0.05867
5	-0.10700	-0.04605	0.00147	
6	-0.25319	-0.13614		
7	-0.44414			

U-SHAPED:

	1	2	3	4
1	1.36143			
2	1.10715	1.03999		
3	0.66670	0.69287	0.76901	
4	0.05017	0.16815	0.39451	0.65915
5	-0.57661	-0.38416	-0.04656	
6	-1.03965	-0.79677		
7	-1.31491			

LAPLACIAN:

	1	2	3	4
1	5.12085			
2	1.97917	1.32350		
3	0.89778	0.61883	0.43783	
4	0.15221	0.18410	0.19675	0.23565
5	-0.54937	-0.19423	0.01378	
6	-1.46048	-0.66484		
7	-2.99848			

NORMAL:

	1	2	3	4
1	2.22030			
2	1.22029	0.83035		
3	0.60903	0.44161	0.34412	
4	0.09848	0.13073	0.16556	0.21045
5	-0.40036	-0.16518	0.00520	
6	-0.96418	-0.49363		
7	-1.78356			

TABLE XVII

Correlationmatrices of the order statistics of the additive noise for a filter length of nine, for six parent distributions.

UNIFORM:

	1	2	3	4	5
1	2.01818				
2	1.52727	1.25455			
3	1.03636	0.87273	0.70909		
4	C.54545	0.49091	0.43636	0.38182	
5	0.05455	0.10909	0.16364	0.21818	0.27273
6	-0.43636	-0.27273	-0.10909	0.05455	
7	-0.92727	-0.65455	-0.38182		
8	-1.41818	-1.03636			
9	-1.90909				

PARABOLIC:

	1	2	3	4	5
1	2.25132				
2	1.56414	1.20129			
3	1.01852	0.79723	0.62197		
4	0.52978	0.44079	0.37266	0.31582	
5	0.06440	0.10444	0.14093	0.17822	0.21921
6	-0.39871	-0.22826	-0.08592	0.04669	
7	-0.87975	-0.57235	-0.31879		
8	-1.40837	-0.94923			
9	-2.05415				

TRIANGULAR:

	1	2	3	4	5
1	0.59606				
2	0.39502	0.29414			
3	0.24882	0.18828	0.14305		
4	0.12713	0.10203	0.08344	0.06869	
5	0.01667	0.02506	0.03166	0.03816	0.04613
6	-0.09251	-0.04998	-0.01770	-0.01048	
7	-0.21016	-0.13015	-0.06968		
8	-0.34834	-0.22399			
9	-0.53165				

U-SHAPED:

	1	2	3	4	5
1	1.42646				
2	1.26020	1.16655			
3	0.99361	0.94320	0.90463		
4	0.57484	0.57705	0.62037	0.69552	
5	0.02877	0.08581	0.20549	0.40315	0.61369
6	-0.52282	-0.41897	-0.23683	0.05225	
7	-0.95422	-0.81691	-0.59016		
8	-1.23311	-1.07389			
9	-1.40748				

LAPLACIAN:

	1	2	3	4	5
1	6.07660				
2	2.96075	1.83018			
3	1.64113	1.01144	0.69665		
4	0.79955	0.51451	0.37308	0.28411	
5	0.13842	0.13920	0.14164	0.15064	0.17507
6	-0.49532	-0.20861	-0.06214	0.04545	
7	-1.24122	-0.60945	-0.28910		
8	-2.31682	-1.18287			
9	-4.44725				

NORMAL:

	1	2	3	4	5
1	2.56255				
2	1.56260	1.09488			
3	0.97012	0.68736	0.51353		
4	0.49898	0.37295	0.29910	0.24592	
5	0.07274	0.09345	0.11377	0.13699	0.16610
6	-0.34819	-0.17939	-0.06366	0.03730	
7	-0.80030	-0.47001	-0.24992		
8	-1.34437	-0.81746			
9	-2.17417				

TABLE XVIII

Correlation matrices of the order statistics of the additive noise for a filter length of eleven, for six parent distributions.

UNIFORM:

	1	2	3	4	5	6
1	2.15385					
2	1.73077	1.46154				
3	1.30769	1.11538	0.92308			
4	0.88462	0.76923	0.65385	0.53846		
5	0.46154	0.42308	0.38462	0.34615	0.30769	
6	0.03846	0.07692	0.11538	0.15385	0.19231	0.23077
7	-0.38462	-0.26923	-0.15385	-0.03846	0.07692	
8	-0.80769	-0.61538	-0.42308	-0.23077		
9	-1.23077	-0.96154	-0.69231			
10	-1.65385	-1.30769				
11	-2.07692					

PARABOLIC:

	1	2	3	4	5	6
1	2.44727					
2	1.80908	1.42981				
3	1.30706	1.03929	0.82703			
4	0.86357	0.69835	0.56822	0.45505		
5	0.44930	0.38208	0.33050	0.28707	0.24926	
6	0.04809	0.07716	0.10284	0.12807	0.15437	0.18316
7	-0.35195	-0.22587	-0.12233	-0.02788	0.06313	
8	-0.76237	-0.53602	-0.35195	-0.18587		
9	-1.19814	-0.86469	-0.59459			
10	-1.68518	-1.23146				
11	-2.28855					

TRIANGULAR:

	1	2	3	4	5	6
1	0.65324					
2	0.46237	0.35486				
3	0.32328	0.24919	0.19365			
4	0.20847	0.16325	0.12938	0.10131		
5	0.10718	0.08825	0.07411	0.06254	0.05298	
6	0.01257	0.01886	0.02366	0.02792	0.03239	0.03791
7	-0.08133	-0.04946	-0.02546	-0.00513	0.01359	
8	-0.18045	-0.12119	-0.07665	-0.03912		
9	-0.29133	-0.20124	-0.13355			
10	-0.42354	-0.29659				
11	-0.59958					

U-SHAPED:

	1	2	3	4	5	6
1	1.46924					
2	1.34688	1.25654				
3	1.17366	1.10404	1.03070			
4	0.90536	0.86412	0.83243	0.81130		
5	0.51083	0.50492	0.52002	0.57170	0.64291	
6	0.01912	0.05057	0.11262	0.22988	0.40436	0.57862
7	-0.47596	-0.41159	-0.30922	-0.13805	0.11391	
8	-0.87842	-0.78948	-0.65710	-0.44600		
9	-1.15493	-1.04938	-0.89635			
10	-1.33391	-1.21698				
11	-1.45950					

LAPLACIAN:

	1	2	3	4	5	6
1	6.97171					
2	3.17004	2.31830				
3	1.96296	1.39780	0.99199			
4	1.17918	0.84554	0.60490	0.44314		
5	0.58866	0.44375	0.33202	0.25766	0.20572	
6	0.08491	0.11079	0.11226	0.11458	0.12157	0.13828
7	-0.40766	-0.20680	-0.09190	-0.01280	0.05348	
8	-0.96215	-0.55845	-0.31389	-0.14703		
9	-1.67304	-1.00594	-0.59402			
10	-2.72224	-1.66522				
11	-4.39070					

NORMAL:

	1	2	3	4	5	6
1	2.85003					
2	1.85000	1.33286				
3	1.26860	0.91428	0.69693			
4	0.81841	0.59773	0.46368	0.36138		
5	0.42562	0.32526	0.26655	0.22377	0.19022	
6	0.05720	0.07192	0.08552	0.10003	0.11674	0.13716
7	-0.30840	-0.17793	-0.09144	-0.01902	0.04862	
8	-0.69165	-0.43863	-0.27482	-0.14088		
9	-1.12114	-0.72971	-0.47845			
10	-1.65523	-1.09056				
11	-2.49341					

TABLE XIX

Correlation matrices of the order statistics of the additive noise for a filter length of thirteen, for six parent distributions.

UNIFORM:

	1	2	3	4
1	2.25714			
2	1.88571	1.62857		
3	1.51429	1.31429	1.11429	
4	1.14286	1.00000	0.85714	0.71429
5	0.77143	0.68571	0.60000	0.51429
6	0.40000	0.37143	0.34286	0.31429
7	0.02857	0.05714	0.08571	0.11429
8	-0.34286	-0.25714	-0.17143	-0.08571
9	-0.71429	-0.57143	-0.42857	-0.28571
10	-1.08571	-0.88571	-0.68571	-0.48571
11	-1.45714	-1.20000	-0.94286	
12	-1.82857	-1.51429		
13	-2.20000			

	5	6	7
5	0.42857		
6	0.28571	0.25714	
7	0.14286	0.17143	0.20000
8	0.00000	0.08571	
9	-0.14286		

PARABOLIC:

	1	2	3	4
1	2.60611			
2	2.00764	1.62418		
3	1.53979	1.24808	1.01757	
4	1.13011	0.92193	0.75778	0.61472
5	0.75174	0.62240	0.52097	0.43310
6	0.39065	0.33762	0.29692	0.26253
7	0.03763	0.05992	0.07921	0.09767
8	-0.31471	-0.21669	-0.13709	-0.06546
9	-0.67364	-0.49805	-0.35665	-0.23052
10	-1.04789	-0.79106	-0.58491	-0.40168
11	-1.45038	-1.10586	-0.82981	
12	-1.90502	-1.46113		
13	-2.47354			

	5	6	7
5	0.35344		
6	0.23231	0.20534	
7	0.11633	0.13594	0.15727
8	0.00240	0.06891	
9	-0.11222		

TRIANGULAR:

	1	2	3	4
1	0.70039			
2	0.51834	0.40773		
3	0.38535	0.30333	0.24182	
4	0.27575	0.21829	0.17521	0.13935
5	0.18000	0.14458	0.11803	0.09596
6	0.09277	0.07786	0.06670	0.05746
7	0.00990	0.01486	0.01859	0.02178
8	-0.07253	-0.04750	-0.02871	-0.01296
9	-0.15846	-0.11227	-0.07760	-0.04860
10	-0.25196	-0.18261	-0.13057	-0.08707
11	-0.35803	-0.26234	-0.19054	
12	-0.48507	-0.35782		
13	-0.65440			

	5	6	7
5	0.07678		
6	0.04956	0.04287	
7	0.02480	0.02808	0.03213
8	0.00112	0.01448	
9	-0.02289		

U-SHAPED:

	1	2	3	4
1	1.49917			
2	1.40148	1.32057		
3	1.27710	1.20706	1.12938	
4	1.09974	1.04502	0.98843	0.93027
5	0.83434	0.80070	0.77092	0.75260
6	0.46267	0.45503	0.45683	0.48077
7	0.01393	0.03386	0.06821	0.13339
8	-0.43703	-0.39232	-0.32918	-0.22841
9	-0.81406	-0.75018	-0.66489	-0.53684
10	-1.08531	-1.00803	-0.90714	-0.75960
11	-1.26709	-1.18056	-1.06873	
12	-1.39409	-1.30066		
13	-1.49318			

	5	6	7
5	0.74260		
6	0.53623	0.60282	
7	0.24640	0.40209	0.55040
8	-0.06793	0.15491	
9	-0.34028		

LAPLACIAN:

	1	2	3	4
1	7.72000			
2	4.26117	2.78203		
3	2.77070	1.78848	1.29513	
4	1.83077	1.18113	0.85490	0.63764
5	1.14138	0.74395	0.54437	0.41164
6	0.58494	0.39631	0.30157	0.23882
7	0.09289	0.09324	0.09336	0.09396
8	-0.38945	-0.20009	-0.10510	-0.04113
9	-0.91531	-0.51702	-0.31718	-0.18316
10	-1.54638	-0.89562	-0.56910	-0.35043
11	-2.37702	-1.39314	-0.89946	
12	-3.61942	-2.13699		
13	-6.09542			

	5	6	7
5	0.31303		
6	0.19302	0.15929	
7	0.09604	0.10162	0.11377
8	0.00892	0.05402	
9	-0.08005		

NORMAL:

	1	2	3	4
1	3.09730			
2	2.09737	1.54549		
3	1.52339	1.11948	0.87361	
4	1.08640	0.80149	0.62859	0.49644
5	0.71318	0.53292	0.42447	0.34235
6	0.37261	0.28960	0.24121	0.20584
7	0.04688	0.05805	0.06793	0.07802
8	-0.27718	-0.17147	-0.10299	-0.04714
9	-0.61230	-0.40813	-0.27857	-0.17494
10	-0.97461	-0.66340	-0.46736	-0.31170
11	-1.39065	-0.95596	-0.68315	
12	-1.91876	-1.32667		
13	-2.76369			

	5	6	7
5	0.27405		
6	0.17772	0.15462	
7	0.08904	0.10168	0.11680
8	0.00337	0.05212	
9	-0.08317		

TABLE XX

Correlation matrices of the order statistics of the additive noise for a filter length of fifteen, for six parent distributions.

UNIFORM:

	1	2	3	4	5
1	2.33824				
2	2.00735	1.76471			
3	1.67647	1.47794	1.27941		
4	1.34559	1.19118	1.03676	0.88235	
5	1.01471	0.90441	0.79412	0.68382	0.57353
6	0.68382	0.61765	0.55147	0.48529	0.41912
7	0.35294	0.33088	0.30882	0.28676	0.26471
8	0.02206	0.04412	0.06618	0.08824	0.11029
9	-0.30882	-0.24265	-0.17647	-0.11029	-0.04412
10	-0.63971	-0.52941	-0.41912	-0.30882	-0.19853
11	-0.97059	-0.81618	-0.66176	-0.50735	-0.35294
12	-1.30147	-1.10294	-0.90441	-0.70588	
13	-1.63235	-1.38971	-1.14706		
14	-1.96324	-1.67647			
15	-2.29412				

	6	7	8
6	0.35294		
7	0.24265	0.22059	
8	0.13235	0.15441	0.17647
9	0.02206	0.08824	
10	-0.08824		

PARABOLIC:

	1	2	3	4	5
1	2.73815				
2	2.17269	1.79082			
3	1.73264	1.42857	1.18913		
4	1.34965	1.11592	0.93205	0.77209	
5	0.99860	0.83072	0.69891	0.58451	0.48048
6	0.66666	0.56190	0.48003	0.40933	0.34541
7	0.34589	0.30270	0.26956	0.24153	0.21679
8	0.03046	0.04823	0.06337	0.07761	0.09171
9	-0.28455	-0.20556	-0.14195	-0.08525	-0.03213
10	-0.60400	-0.46268	-0.34970	-0.24972	-0.15685
11	-0.93350	-0.72766	-0.56357	-0.41880	-0.28475
12	-1.28055	-1.00656	-0.78847	-0.59636	
13	-1.65695	-1.30886	-1.03204		
14	-2.08526	-1.65266			
15	-2.62447				

	6	7	8
6	0.28612		
7	0.19452	0.17432	
8	0.10615	0.12135	0.13779
9	0.01920	0.07001	
10	-0.06791		

TRIANGULAR:

	1	2	3	4	5
1	0.74014				
2	0.56580	0.45397			
3	0.43817	0.35133	0.28621		
4	0.33296	0.26757	0.21854	0.17768	
5	0.24133	0.19507	0.16038	0.13148	0.10624
6	0.15868	0.12998	0.10846	0.09054	0.07491
7	0.08185	0.06973	0.06064	0.05308	0.04653
8	0.00806	0.01209	0.01512	0.01766	0.01996
9	-0.06545	-0.04513	-0.02988	-0.01715	-0.00592
10	-0.14144	-0.10412	-0.07612	-0.05276	-0.03222
11	-0.22267	-0.16709	-0.12539	-0.09062	-0.06006
12	-0.31221	-0.23645	-0.17962	-0.13223	
13	-0.41428	-0.31550	-0.24141		
14	-0.53670	-0.41030			
15	-0.69991				

	6	7	8
6	0.06096		
7	0.04080	0.03588	
8	0.02223	0.02475	0.02785
9	0.00444	0.01447	
10	-0.01344		

U-SHAPED:

	1	2	3	4	5
1	1.52121				
2	1.43923	1.36759			
3	1.34192	1.27671	1.20403		
4	1.21556	1.15907	1.09763	1.03036	
5	1.03597	0.99179	0.94547	0.89942	0.85281
6	0.77567	0.74812	0.72154	0.70090	0.69227
7	0.42466	0.41733	0.41415	0.42218	0.45213
8	0.01075	0.02482	0.04597	0.08295	0.14982
9	-0.40470	-0.37111	-0.32801	-0.26535	-0.16654
10	-0.75953	-0.71040	-0.64992	-0.56702	-0.44319
11	-1.02419	-0.96385	-0.89081	-0.79320	-0.65094
12	-1.20727	-1.13909	-1.05718	-0.94900	
13	-1.33585	-1.26188	-1.17337		
14	-1.43439	-1.35575			
15	-1.51706				

	6	7	8
6	0.68962		
7	0.50886	0.57089	
8	0.25784	0.39813	0.52698
9	-0.01608	0.18338	
10	-0.25987		

LAPLACIAN:

	1	2	3	4	5
1	8.47150				
2	4.11305	3.22133			
3	2.82697	2.14364	1.59627		
4	1.98592	1.49649	1.10988	0.84633	
5	1.36326	1.02992	0.76578	0.58583	0.45100
6	0.86564	0.66344	0.49887	0.38664	0.30273
7	0.44182	0.35576	0.27726	0.22345	0.18346
8	0.05492	0.07872	0.07991	0.08010	0.08072
9	-0.32772	-0.19190	-0.11090	-0.05670	-0.01543
10	-0.73846	-0.47976	-0.31231	-0.19968	-0.11443
11	-1.21278	-0.81048	-0.54272	-0.36231	-0.22606
12	-1.79753	-1.21734	-0.82567	-0.56156	
13	-2.57409	-1.75732	-1.20099		
14	-3.72435	-2.55743			
15	-5.38970				

	6	7	8
6	0.23639		
7	0.15254	0.12901	
8	0.08260	0.08714	0.09635
9	0.01930	0.05207	
10	-0.04410		

NORMAL:

	1	2	3	4	5
1	3.31444				
2	2.31440	1.73646			
3	1.74580	1.30507	1.03885		
4	1.31800	0.98603	0.78570	0.63328	
5	0.95778	0.71995	0.57688	0.46840	0.37782
6	0.63469	0.48277	0.39208	0.32387	0.26743
7	0.33225	0.26169	0.22071	0.19076	0.16683
8	0.03957	0.04842	0.05601	0.06352	0.07143
9	-0.25204	-0.16355	-0.10720	-0.06207	-0.02212
10	-0.55108	-0.38051	-0.27386	-0.18988	-0.11683
11	-0.86768	-0.60985	-0.44968	-0.32434	-0.21603
12	-1.21654	-0.86221	-0.64283	-0.47170	
13	-1.62360	-1.15632	-0.86761		
14	-2.14775	-1.53462			
15	-2.99821				

	6	7	8
6	0.21829		
7	0.14689	0.13002	
8	0.08014	0.09005	0.10169
9	0.01543	0.05235	
10	-0.04944		

TABLE XXI

Correlation matrices of the order statistics of the additive noise for a filter length of seventeen, for six parent distributions.

UNIFORM:

	1	2	3	4	5	6
1	2.40351					
2	2.10526	1.87719				
3	1.80702	1.61404	1.42105			
4	1.50877	1.35088	1.19298	1.03509		
5	1.21053	1.08772	0.96491	0.84211	0.71930	
6	0.91228	0.82456	0.73684	0.64912	0.56140	0.47368
7	0.61404	0.56140	0.50877	0.45614	0.40351	0.35088
8	0.31579	0.29825	0.28070	0.26316	0.24561	0.22807
9	0.01754	0.03509	0.05263	0.07018	0.08772	0.10526
10	-0.28070	-0.22807	-0.17544	-0.12281	-0.07018	-0.01754
11	-0.57895	-0.49123	-0.40351	-0.31579	-0.22807	-0.14035
12	-0.87719	-0.75439	-0.63158	-0.50877	-0.38596	-0.26316
13	-1.17544	-1.01754	-0.85965	-0.70175	-0.54386	
14	-1.47368	-1.28070	-1.08772	-0.89474		
15	-1.77193	-1.54386	-1.31579			
16	-2.07018	-1.80702				
17	-2.36842					

	7	8	9
7	0.29825		
8	0.21053	0.19298	
9	0.12281	0.14035	0.15789
10	0.03509	0.08772	
11	-0.05263		

PARABOLIC:

	1	2	3	4	5	6
1	2.85015					
2	2.31268	1.93523				
3	1.89585	1.58588	1.34254			
4	1.53472	1.28540	1.08978	0.92005		
5	1.20547	1.01262	0.86145	0.73042	0.61141	
6	0.89615	0.75706	0.64822	0.55406	0.46873	0.38929
7	0.59952	0.51245	0.44460	0.38616	0.33346	0.28466
8	0.31054	0.27450	0.24686	0.22347	0.20279	0.18409
9	0.02530	0.03988	0.05216	0.06356	0.07468	0.08586
10	-0.25967	-0.19430	-0.14198	-0.09567	-0.05264	-0.01145
11	-0.54778	-0.43089	-0.33796	-0.25622	-0.18080	-0.10916
12	-0.84287	-0.67307	-0.53841	-0.42028	-0.31158	-0.20864
13	-1.14974	-0.92479	-0.74663	-0.59055	-0.44716	
14	-1.47520	-1.19164	-0.96725	-0.77084		
15	-1.83032	-1.48268	-1.20775			
16	-2.23661	-1.81554				
17	-2.75073					

	7	8	9
7	0.23873		
8	0.16699	0.15129	
9	0.09740	0.10954	0.12260
10	0.02884	0.06895	
11	-0.03967		

TRIANGULAR:

	1	2	3	4	5	6
1	0.77427					
2	0.60672	0.49473				
3	0.48384	0.39407	0.32675			
4	0.38248	0.31177	0.25874	0.21454		
5	0.29428	0.24052	0.20020	0.16660	0.13722	
6	0.21501	0.17673	0.14801	0.12409	0.10317	0.08437
7	0.14205	0.11819	0.10029	0.08538	0.07235	0.06067
8	0.07328	0.06317	0.05559	0.04928	0.04378	0.03889
9	0.00672	0.01008	0.01260	0.01471	0.01658	0.01833
10	-0.05964	-0.04272	-0.03002	-0.01944	-0.01015	-0.00170
11	-0.12784	-0.09687	-0.07364	-0.05428	-0.03730	-0.02193
12	-0.19985	-0.15399	-0.11959	-0.09091	-0.06578	-0.04305
13	-0.27776	-0.21574	-0.16922	-0.13045	-0.09648	
14	-0.36406	-0.28413	-0.22418	-0.17422		
15	-0.46261	-0.36222	-0.28694			
16	-0.58085	-0.45592				
17	-0.73850					

	7	8	9
7	0.05007		
8	0.03454	0.03077	
9	0.02011	0.02211	0.02455
10	0.00625	0.01406	
11	-0.00758		

U-SHAPED:

	1	2	3	4	5	6
1	1.53811					
2	1.46715	1.40337				
3	1.38640	1.32686	1.26105			
4	1.28894	1.23482	1.17557	1.10992		
5	1.16081	1.11410	1.06362	1.00954	0.95092	
6	0.98040	0.94397	0.90546	0.86625	0.82880	0.79112
7	0.72624	0.70341	0.68044	0.65982	0.64650	0.64498
8	0.39367	0.38715	0.38251	0.38322	0.39623	0.43025
9	0.00863	0.01931	0.03379	0.05633	0.09549	0.16298
10	-0.37755	-0.35099	-0.31902	-0.27672	-0.21429	-0.11778
11	-0.71295	-0.67346	-0.62732	-0.56908	-0.48798	-0.36833
12	-0.97046	-0.92140	-0.86475	-0.79468	-0.69965	-0.56256
13	-1.15370	-1.09781	-1.03364	-0.95504	-0.84979	
14	-1.28373	-1.22281	-1.15308	-1.06809		
15	-1.38230	-1.31739	-1.24322			
16	-1.46367	-1.39536				
17	-1.53502					

	7	8	9
7	0.64732		
8	0.48686	0.54465	
9	0.26586	0.39335	0.50709
10	0.02346	0.20379	
11	-0.19696		

LAPLACIAN:

	1	2	3	4	5	6
1	9.13055					
2	4.51536	3.63774				
3	3.20236	2.49398	1.89134			
4	2.33961	1.80812	1.36442	1.06043		
5	1.70040	1.31273	0.99031	0.76963	0.60403	
6	1.19217	0.92494	0.70042	0.54671	0.43140	0.33938
7	0.76678	0.60413	0.46252	0.36537	0.29255	0.23459
8	0.39296	0.32516	0.25722	0.21024	0.17508	0.14734
9	0.04549	0.06855	0.06982	0.06990	0.07009	0.07071
10	-0.29912	-0.18354	-0.11295	-0.06580	-0.03027	-0.00126
11	-0.66428	-0.44872	-0.30414	-0.20682	-0.13360	-0.07433
12	-1.07466	-0.74545	-0.51738	-0.36346	-0.24774	-0.15432
13	-1.55981	-1.09553	-0.76856	-0.54764	-0.38159	
14	-2.16340	-1.53076	-1.08067	-0.77634		
15	-2.96698	-2.11010	-1.49607			
16	-4.15373	-2.96611				
17	-5.78851					

	7	8	9
7	0.18697		
8	0.12514	0.10787	
9	0.07240	0.07617	0.08338
10	0.02436	0.04927	
11	-0.02316		

NORMAL:

	1	2	3	4	5	6
1	3.50776					
2	2.50772	1.90933				
3	1.94324	1.47382	1.19221			
4	1.52226	1.15396	0.33305	0.76588		
5	1.17138	0.88963	0.72085	0.59330	0.48713	
6	0.86038	0.65661	0.53492	0.44323	0.36715	0.30058
7	0.57336	0.44236	0.36468	0.30656	0.25871	0.21721
8	0.30035	0.23914	0.20371	0.17785	0.15715	0.13980
9	0.03414	0.04139	0.04746	0.05331	0.05932	0.06574
10	-0.23135	-0.15549	-0.10782	-0.07014	-0.03731	-0.00699
11	-0.50210	-0.35599	-0.26569	-0.19540	-0.13506	-0.08021
12	-0.78493	-0.56521	-0.43021	-0.32571	-0.23648	-0.15589
13	-1.08899	-0.78991	-0.60670	-0.46528	-0.34489	
14	-1.42841	-1.04053	-0.80335	-0.62058		
15	-1.82902	-1.33609	-1.03506			
16	-2.35033	-1.72042				
17	-3.20541					

	7	8	9
7	0.18004		
8	0.12491	0.11205	
9	0.07282	0.08080	0.09005
10	0.02217	0.05115	
11	-0.02837		

AD-A124 459 NONLINEAR FILTERING USING LINEAR COMBINATIONS OF ORDER STATISTICS(U) ILLINOIS UNIV AT URBANA COORDINATED SCIENCE LAB A C BOVIK JAN 82 R-935 N00014-79-C-0424 2/2

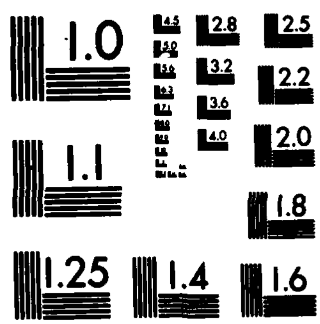
UNCLASSIFIED F/G 12/1 NL

END

DATE FILMED

83

DTIC



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

TABLE XXII

Correlationmatrices of the order statistics of the additive noise for a filter length of nineteen, for six parent distributions.

UNIFORM:

	1	2	3	4	5	6
1	2.45714					
2	2.18571	1.97143				
3	1.91428	1.72857	1.54286			
4	1.64285	1.48571	1.32857	1.17143		
5	1.37143	1.24286	1.11428	0.98571	0.85714	
6	1.10000	1.00000	0.90000	0.80000	0.70000	0.60000
7	0.82857	0.75714	0.68571	0.61429	0.54286	0.47143
8	0.55714	0.51429	0.47143	0.42857	0.38571	0.34286
9	0.28571	0.27143	0.25714	0.24286	0.22857	0.21429
10	0.01429	0.02857	0.04286	0.05714	0.07143	0.08571
11	-0.25714	-0.21429	-0.17143	-0.12857	-0.08571	-0.04286
12	-0.52857	-0.45714	-0.38571	-0.31429	-0.24286	-0.17143
13	-0.80000	-0.70000	-0.60000	-0.50000	-0.40000	-0.30000
14	-1.07143	-0.94286	-0.81428	-0.68571	-0.55714	-0.42857
15	-1.34286	-1.18571	-1.02857	-0.87143	-0.71428	
16	-1.61428	-1.42857	-1.24286	-1.05714		
17	-1.88571	-1.67143	-1.45714			
18	-2.15714	-1.91428				
19	-2.42857					

	7	8	9	10
7	0.40000			
8	0.30000	0.25714		
9	0.20000	0.18571	0.17143	
10	0.10000	0.11429	0.12857	0.14286
11	0.00000	0.04286	0.08571	
12	-0.10000	-0.02857		
13	-0.20000			

PARABOLIC:

	1	2	3	4	5	6
1	2.94670					
2	2.43337	2.06175				
3	2.03634	1.72422	1.47988			
4	1.69356	1.43472	1.23214	1.05679		
5	1.38232	1.17285	1.00899	0.86724	0.73876	
6	1.09129	0.92859	0.80143	0.69153	0.59202	0.49944
7	0.81372	0.69603	0.60420	0.52497	0.45336	0.38688
8	0.54505	0.47122	0.41382	0.36449	0.32009	0.27906
9	0.28189	0.25123	0.22774	0.20785	0.19026	0.17431
10	0.02144	0.03368	0.04388	0.05327	0.06231	0.07130
11	-0.23882	-0.18357	-0.13959	-0.10085	-0.06508	-0.03108
12	-0.50140	-0.40264	-0.32448	-0.25604	-0.19323	-0.13392
13	-0.76902	-0.62583	-0.51274	-0.41397	-0.32352	-0.23833
14	-1.04499	-0.85588	-0.70671	-0.57658	-0.45758	-0.34565
15	-1.33365	-1.09642	-0.90944	-0.74646	-0.59753	
16	-1.64135	-1.35276	-1.12542	-0.92735		
17	-1.97861	-1.63365	-1.36200			
18	-2.36610	-1.95627				
19	-2.85843					

	7	8	9	10
7	0.32406			
8	0.24050	0.20384		
9	0.15966	0.14611	0.13356	
10	0.08043	0.08987	0.09980	0.11042
11	0.00190	0.03441	0.06690	
12	-0.07679	-0.02092		
13	-0.15653			

TRIANGULAR:

	1	2	3	4	5	6
1	0.80398					
2	0.64247	0.53097				
3	0.52385	0.43234	0.36371			
4	0.42594	0.35158	0.29581	0.24933		
5	0.34073	0.28161	0.23728	0.20033	0.16801	
6	0.26426	0.21902	0.18509	0.15682	0.13209	0.10984
7	0.19414	0.16177	0.13749	0.11726	0.09956	0.08365
8	0.12867	0.10843	0.09325	0.08060	0.06953	0.05959
9	0.06636	0.05777	0.05133	0.04596	0.04127	0.03707
10	0.00572	0.00857	0.01072	0.01250	0.01408	0.01551
11	-0.05479	-0.04042	-0.02964	-0.02065	-0.01278	-0.00567
12	-0.11670	-0.09046	-0.07079	-0.05439	-0.04003	-0.02708
13	-0.18150	-0.14280	-0.11378	-0.08959	-0.06842	-0.04932
14	-0.25067	-0.19864	-0.15962	-0.12710	-0.09863	-0.07297
15	-0.32588	-0.25934	-0.20944	-0.16785	-0.13144	
16	-0.40936	-0.32671	-0.26472	-0.21306		
17	-0.50474	-0.40368	-0.32789			
18	-0.61919	-0.49604				
19	-0.77179					

	7	8	9	10
7	0.06909			
8	0.05052	0.04220		
9	0.03328	0.02987	0.02690	
10	0.01690	0.01834	0.01996	0.02194
11	0.00093	0.00724	0.01350	
12	-0.01511	-0.00379		
13	-0.03171			

U-SHAPED:

	1	2	3	4	5	6
1	1.55147					
2	1.48877	1.43146				
3	1.41922	1.36496	1.30561			
4	1.33923	1.28865	1.23354	1.17274		
5	1.24131	1.19546	1.14578	1.09172	1.03216	
6	1.11176	1.07235	1.03002	0.98477	0.93706	0.88569
7	0.93154	0.90092	0.86852	0.83490	0.80178	0.77139
8	0.68392	0.66473	0.64507	0.62605	0.61050	0.60309
9	0.36781	0.36211	0.35734	0.35512	0.35922	0.37620
10	0.00711	0.01562	0.02638	0.04146	0.06542	0.10630
11	-0.35443	-0.33271	-0.30758	-0.27664	-0.23468	-0.17248
12	-0.67272	-0.64003	-0.60303	-0.55903	-0.50232	-0.42277
13	-0.92297	-0.88196	-0.83596	-0.78208	-0.71418	-0.62138
14	-1.10552	-1.05847	-1.00593	-0.94482	-0.86868	-0.76599
15	-1.23672	-1.18521	-1.12782	-1.06133	-0.97898	
16	-1.33565	-1.28064	-1.21944	-1.14867		
17	-1.41622	-1.35827	-1.29383			
18	-1.48613	-1.42557				
19	-1.54907					

	7	8	9	10
7	0.74073			
8	0.60685	0.61262		
9	0.41290	0.46860	0.52256	
10	0.17366	0.27150	0.38822	0.48990
11	-0.07849	0.05435	0.21876	
12	-0.30726	-0.14660		
13	-0.48934			

LAPLACIAN:

	1	2	3	4	5	6
1	9.74235					
2	4.88036	4.03317				
3	3.54638	2.82903	2.17858			
4	2.66589	2.10892	1.61545	1.27553		
5	2.01246	1.58767	1.21444	0.95770	0.76493	
6	1.49340	1.17972	0.90338	0.71334	0.57067	0.45659
7	1.06171	0.84392	0.64900	0.51483	0.41411	0.33362
8	0.68879	0.55631	0.43233	0.34675	0.28251	0.23123
9	0.35369	0.29997	0.24029	0.19866	0.16742	0.14257
10	0.03817	0.06058	0.06196	0.06201	0.06207	0.06227
11	-0.27537	-0.17549	-0.11296	-0.07122	-0.03988	-0.01460
12	-0.60449	-0.42181	-0.29471	-0.20900	-0.14465	-0.09294
13	-0.96724	-0.69227	-0.49374	-0.35943	-0.25859	-0.17765
14	-1.38385	-1.00230	-0.72159	-0.53138	-0.38857	-0.27402
15	-1.88102	-1.37199	-0.99314	-0.73618	-0.54327	
16	-2.50151	-1.83328	-1.33191	-0.99163		
17	-3.32811	-2.44777	-1.78319			
18	-4.54426	-3.35235				
19	-6.13536					

	7	8	9	10
7	0.26676			
8	0.18878	0.15298		
9	0.12222	0.10555	0.09239	
10	0.06289	0.06440	0.06758	0.07338
11	0.00699	0.02672	0.04631	
12	-0.04920	-0.01024		
13	-0.10944			

NORMAL:

	1	2	3	4	5	6
1	3.68204					
2	2.68199	2.06701				
3	2.12079	1.62824	1.33451			
4	1.70511	1.30773	1.07070	0.89222		
5	1.36132	1.04468	0.85592	0.71386	0.59609	
6	1.05926	0.81469	0.66911	0.55970	0.46912	0.39000
7	0.78328	0.60528	0.49962	0.42044	0.35508	0.29818
8	0.52386	0.40891	0.34113	0.29064	0.24925	0.21348
9	0.27445	0.22048	0.18935	0.16666	0.14850	0.13323
10	0.02996	0.03605	0.04104	0.04576	0.05051	0.05549
11	-0.21402	-0.14778	-0.10659	-0.07439	-0.04664	-0.02135
12	-0.46185	-0.33432	-0.25623	-0.19600	-0.14480	-0.09877
13	-0.71841	-0.52727	-0.41086	-0.32152	-0.24594	-0.17836
14	-0.98982	-0.73124	-0.57419	-0.45397	-0.35252	-0.26205
15	-1.28477	-0.95275	-0.75144	-0.59756	-0.46792	
16	-1.61716	-1.20223	-0.95094	-0.75906		
17	-2.01287	-1.49908	-1.18817			
18	-2.53205	-1.88835				
19	-3.39107					

	7	8	9	10
7	0.24693			
8	0.18155	0.15239		
9	0.12005	0.10850	0.09838	
10	0.06083	0.06670	0.07327	0.08079
11	0.00260	0.02596	0.04933	
12	-0.05582	-0.01459		
13	-0.11565			

TABLE XXIII

Correlation matrices of the order statistics of the additive noise for a filter length of twenty-one, for six parent distributions.

UNIFORM:

	1	2	3	4	5	6
1	2.50198					
2	2.25297	2.05138				
3	2.00395	1.82609	1.64822			
4	1.75494	1.60079	1.44664	1.29249		
5	1.50593	1.37549	1.24506	1.11462	0.98419	
6	1.25692	1.15020	1.04348	0.93676	0.83004	0.72332
7	1.00790	0.92490	0.84190	0.75889	0.67589	0.59289
8	0.75889	0.69960	0.64032	0.58103	0.52174	0.46245
9	0.50988	0.47431	0.43874	0.40316	0.36759	0.33202
10	0.26087	0.24901	0.23715	0.22530	0.21344	0.20158
11	0.01186	0.02372	0.03557	0.04743	0.05929	0.07115
12	-0.23715	-0.20158	-0.16601	-0.13043	-0.09486	-0.05929
13	-0.48617	-0.42688	-0.36759	-0.30830	-0.24901	-0.18972
14	-0.73518	-0.65217	-0.56917	-0.48617	-0.40316	-0.32016
15	-0.98419	-0.87747	-0.77075	-0.66403	-0.55731	-0.45059
16	-1.23320	-1.10277	-0.97233	-0.84190	-0.71146	-0.58103
17	-1.48221	-1.32806	-1.17391	-1.01976	-0.86561	
18	-1.73123	-1.55336	-1.37549	-1.19763		
19	-1.98024	-1.77866	-1.57707			
20	-2.22925	-2.00395				
21	-2.47826					

	7	8	9	10	11
7	0.50988				
8	0.40316	0.34387			
9	0.29644	0.26087	0.22530		
10	0.18972	0.17787	0.16601	0.15415	
11	0.08300	0.09486	0.10672	0.11858	0.13043
12	-0.02372	0.01186	0.04743	0.08300	
13	-0.13043	-0.07115	-0.01186		
14	-0.23715	-0.15415			
15	-0.34387				

PARABOLIC:

	1	2	3	4	5	6
1	3.03108					
2	2.53884	2.17369				
3	2.15896	1.84699	1.60335			
4	1.83190	1.56739	1.36084	1.18246		
5	1.53589	1.31521	1.14293	0.99419	0.85964	
6	1.26009	1.08077	0.94085	0.82011	0.71095	0.60955
7	0.99813	0.85844	0.74954	0.65564	0.57083	0.49212
8	0.74577	0.64451	0.56569	0.49783	0.43664	0.37996
9	0.49992	0.43629	0.38693	0.34457	0.30652	0.27142
10	0.25818	0.23169	0.21140	0.19424	0.17904	0.16525
11	0.01847	0.02892	0.03757	0.04547	0.05301	0.06043
12	-0.22111	-0.17363	-0.13598	-0.10298	-0.07264	-0.04396
13	-0.46244	-0.37759	-0.31066	-0.25230	-0.19894	-0.14878
14	-0.70755	-0.58467	-0.48795	-0.40377	-0.32699	-0.25497
15	-0.95881	-0.79687	-0.66957	-0.55889	-0.45804	-0.36357
16	-1.21919	-1.01672	-0.85766	-0.71948	-0.59366	-0.47589
17	-1.49272	-1.24762	-1.05516	-0.88804	-0.73594	
18	-1.78543	-1.49466	-1.26642	-1.06829		
19	-2.10741	-1.76635	-1.49870			
20	-2.47858	-2.07947				
21	-2.95174					

	7	8	9	10	11
7	0.41777				
8	0.32652	0.27552			
9	0.23847	0.20719	0.17723		
10	0.15255	0.14075	0.12975	0.11949	
11	0.06788	0.07548	0.08336	0.09164	0.10045
12	-0.01630	0.01075	0.03757	0.06443	
13	-0.10073	-0.05402	-0.00808		
14	-0.18614	-0.11943			
15	-0.27341				

TRIANGULAR:

	1	2	3	4	5	6
1	0.83017					
2	0.67408	0.56344				
3	0.55928	0.46683	0.39749			
4	0.46448	0.38761	0.32996	0.28191		
5	0.38196	0.31895	0.27168	0.23230	0.19784	
6	0.30793	0.25751	0.21970	0.18819	0.16062	0.13581
7	0.24015	0.20139	0.17231	0.14809	0.12689	0.10782
8	0.17713	0.14929	0.12841	0.11101	0.09579	0.08209
9	0.11767	0.10022	0.08713	0.07622	0.06667	0.05809
10	0.06065	0.05324	0.04767	0.04304	0.03899	0.03534
11	0.00494	0.00740	0.00925	0.01080	0.01215	0.01338
12	-0.05068	-0.03828	-0.02897	-0.02122	-0.01443	-0.00832
13	-0.10739	-0.08480	-0.06786	-0.05374	-0.04138	-0.03025
14	-0.16636	-0.13314	-0.10822	-0.08746	-0.06929	-0.05292
15	-0.22871	-0.18422	-0.15086	-0.12305	-0.09872	-0.07681
16	-0.29559	-0.23901	-0.19658	-0.16121	-0.13027	-0.10240
17	-0.36847	-0.29871	-0.24638	-0.20278	-0.16462	
18	-0.44941	-0.36501	-0.30170	-0.24895		
19	-0.54191	-0.44077	-0.36492			
20	-0.65291	-0.53169				
21	-0.80091					

	7	8	9	10	11
7	0.09034				
8	0.06955	0.05794			
9	0.05024	0.04299	0.03628		
10	0.03203	0.02900	0.02627	0.02386	
11	0.01452	0.01565	0.01684	0.01819	0.01982
12	-0.00268	0.00263	0.00776	0.01288	
13	-0.02001	-0.01042	-0.00124		
14	-0.03788	-0.02381			
15	-0.05668				

U-SHAPED:

	1	2	3	4	5	6
1	1.56232					
2	1.50610	1.45410				
3	1.44473	1.39507	1.34129			
4	1.37626	1.32929	1.27850	1.22297		
5	1.29667	1.25296	1.20582	1.15458	1.09818	
6	1.19813	1.15864	1.11622	1.07045	1.02099	0.96675
7	1.06752	1.03372	0.99765	0.95916	0.91853	0.87635
8	0.88821	0.86208	0.83448	0.80558	0.77624	0.74836
9	0.64723	0.63086	0.61398	0.59706	0.58150	0.57031
10	0.34582	0.34085	0.33638	0.33318	0.33315	0.34020
11	0.00599	0.01298	0.02145	0.03242	0.04821	0.07351
12	-0.33450	-0.31630	-0.29577	-0.27168	-0.24134	-0.19947
13	-0.63762	-0.60996	-0.57930	-0.54422	-0.50177	-0.44616
14	-0.88073	-0.84576	-0.80725	-0.76368	-0.71188	-0.64564
15	-1.06197	-1.02159	-0.97730	-0.92745	-0.86869	-0.79449
16	-1.19402	-1.14963	-1.10103	-1.04649	-0.98250	-0.90223
17	-1.29348	-1.24597	-1.19399	-1.13576	-1.06764	
18	-1.37361	-1.32350	-1.26871	-1.20739		
19	-1.44242	-1.39002	-1.33275			
20	-1.50401	-1.44953				
21	-1.56039					

	7	8	9	10	11
7	0.83110				
8	0.72380	0.69873			
9	0.56767	0.57537	0.58354		
10	0.36031	0.39873	0.45310	0.50360	
11	0.11568	0.18240	0.27545	0.38299	0.47483
12	-0.13760	-0.04626	0.07895	0.22993	
13	-0.36805	-0.25660	-0.10557		
14	-0.55491	-0.42776			
15	-0.69417				

LAPLACIAN:

	1	2	3	4	5	6
1	10.31399					
2	5.21304	4.40956				
3	3.86277	3.14941	2.45735			
4	2.96782	2.39873	1.86148	1.48921		
5	2.30242	1.85421	1.43614	1.14698	0.92978	
6	1.77364	1.42775	1.10575	0.88317	0.71600	0.58227
7	1.33470	1.07719	0.83570	0.66872	0.54332	0.44301
8	0.95818	0.77871	0.60680	0.48777	0.39837	0.32687
9	0.62539	0.51663	0.40664	0.33019	0.27277	0.22687
10	0.32136	0.27878	0.22574	0.18840	0.16035	0.13795
11	0.03233	0.05417	0.05565	0.05571	0.05572	0.05579
12	-0.25529	-0.16793	-0.11184	-0.07439	-0.04631	-0.02377
13	-0.55505	-0.39824	-0.28496	-0.20839	-0.15093	-0.10489
14	-0.88069	-0.64759	-0.47197	-0.35280	-0.26337	-0.19173
15	-1.24686	-0.92748	-0.68165	-0.51451	-0.38906	-0.28861
16	-1.67140	-1.25172	-0.92442	-0.70164	-0.53442	-0.40054
17	-2.17982	-1.63992	-1.21504	-0.92560	-0.70635	
18	-2.81503	-2.12489	-1.57808	-1.20536		
19	-3.66127	-2.77099	-2.06173			
20	-4.90096	-3.71804				
21	-6.43833					

	7	8	9	10	11
7	0.35949				
8	0.26738	0.21659			
9	0.18873	0.15630	0.12844		
10	0.11942	0.10387	0.09094	0.08060	
11	0.05601	0.05660	0.05796	0.06068	0.06544
12	-0.00481	0.01194	0.02763	0.04345	
13	-0.06630	-0.03264	-0.00195		
14	-0.13177	-0.07969			
15	-0.20459				

NORMAL:

	1	2	3	4	5	6
1	3.84073					
2	2.84067	2.21188				
3	2.28214	1.77042	1.46685			
4	1.87072	1.44938	1.19932	1.01187		
5	1.53251	1.18732	0.98249	0.82897	0.70220	
6	1.23736	0.95965	0.79498	0.67164	0.56987	0.48125
7	0.96973	0.75385	0.62602	0.53041	0.45162	0.38312
8	0.72032	0.56249	0.46929	0.39977	0.34262	C.29309
9	0.48292	0.38066	0.32064	0.27612	0.23974	0.20841
10	0.25294	0.20475	0.17704	0.15687	0.14074	0.12716
11	0.02665	0.03186	0.03606	0.03997	0.04385	0.04784
12	-0.19927	-0.14059	-0.10442	-0.07638	-0.05244	-0.03084
13	-0.42810	-0.31514	-0.24650	-0.19394	-0.14960	-0.11009
14	-0.66346	-0.49454	-0.39242	-0.31458	-0.24920	-0.19121
15	-0.90969	-0.68212	-0.54490	-0.44055	-0.35310	-0.27572
16	-1.17257	-0.88229	-0.70752	-0.57480	-0.46374	-0.36560
17	-1.46056	-1.10147	-0.88550	-0.72165	-0.58467	
18	-1.78748	-1.35018	-1.08737	-0.88812		
19	-2.17933	-1.64817	-1.32913			
20	-2.69686	-2.04157				
21	-3.55929					

	7	8	9	10	11
7	0.32149				
8	0.24866	0.20786			
9	0.18051	0.15511	0.13161		
10	0.11539	0.10503	0.09582	0.08764	
11	0.05204	0.05656	0.06151	0.06702	0.07326
12	-0.01061	0.00885	0.02802	0.04729	
13	-0.07357	-0.03890	-0.00525		
14	-0.13787	-0.08749			
15	-0.20473				

TABLE XXIV

Correlation matrices of the order statistics of the additive noise for a filter length of twenty-three, for six parent distributions.

UNIFORM:

	1	2	3	4	5	6
1	2.54000					
2	2.31000	2.12000				
3	2.08000	1.91000	1.74000			
4	1.85000	1.70000	1.55000	1.40000		
5	1.62000	1.49000	1.36000	1.23000	1.10000	
6	1.39000	1.28000	1.17000	1.06000	0.95000	0.84000
7	1.16000	1.07000	0.98000	0.89000	0.80000	0.71000
8	0.93000	0.86000	0.79000	0.72000	0.65000	0.58000
9	0.70000	0.65000	0.60000	0.55000	0.50000	0.45000
10	0.47000	0.44000	0.41000	0.38000	0.35000	0.32000
11	0.24000	0.23000	0.22000	0.21000	0.20000	0.19000
12	0.01000	0.02000	0.03000	0.04000	0.05000	0.06000
13	-0.22000	-0.19000	-0.16000	-0.13000	-0.10000	-0.07000
14	-0.45000	-0.40000	-0.35000	-0.30000	-0.25000	-0.20000
15	-0.68000	-0.61000	-0.54000	-0.47000	-0.40000	-0.33000
16	-0.91000	-0.82000	-0.73000	-0.64000	-0.55000	-0.46000
17	-1.14000	-1.03000	-0.92000	-0.81000	-0.70000	-0.59000
18	-1.37000	-1.24000	-1.11000	-0.98000	-0.85000	-0.72000
19	-1.60000	-1.45000	-1.30000	-1.15000	-1.00000	
20	-1.83000	-1.66000	-1.49000	-1.32000		
21	-2.06000	-1.87000	-1.68000			
22	-2.29000	-2.08000				
23	-2.52000					

	7	8	9	10	11	12
7	0.62000					
8	0.51000	0.44000				
9	0.40000	0.35000	0.30000			
10	0.29000	0.26000	0.23000	0.20000		
11	0.18000	0.17000	0.16000	0.15000	0.14000	
12	0.07000	0.08000	0.09000	0.10000	0.11000	0.12000
13	-0.04000	-0.01000	0.02000	0.05000	0.08000	
14	-0.15000	-0.10000	-0.05000	0.00000		
15	-0.26000	-0.19000	-0.12000			
16	-0.37000	-0.28000				
17	-0.48000					

PARABOLIC:

	1	2	3	4	5	6
1	3.10563					
2	2.63204	2.27359				
3	2.26722	1.95682	1.71490			
4	1.95383	1.68620	1.47762	1.29785		
5	1.67093	1.44267	1.26482	1.11154	0.97315	
6	1.40811	1.21690	1.06795	0.93963	0.82381	0.71640
7	1.15927	1.00344	0.88212	0.77764	0.68339	0.59603
8	0.92041	0.79877	0.70413	0.62270	0.54930	0.48133
9	0.68868	0.60036	0.53175	0.47280	0.41973	0.37067
10	0.46188	0.40631	0.36327	0.32640	0.29333	0.26286
11	0.23822	0.21503	0.19729	0.18229	0.16899	0.15693
12	0.01612	0.02518	0.03263	0.03939	0.04580	0.05205
13	-0.20588	-0.16451	-0.13183	-0.10327	-0.07712	-0.05250
14	-0.42924	-0.35531	-0.29719	-0.24665	-0.20060	-0.15747
15	-0.65551	-0.54855	-0.46461	-0.39176	-0.32552	-0.26360
16	-0.88645	-0.74572	-0.63540	-0.53975	-0.45286	-0.37173
17	-1.12420	-0.94866	-0.81114	-0.69199	-0.58382	-0.48289
18	-1.37150	-1.15971	-0.99387	-0.85024	-0.71991	-0.59835
19	-1.63217	-1.38215	-1.18641	-1.01696	-0.86323	
20	-1.91199	-1.62088	-1.39303	-1.19582		
21	-2.22068	-1.88419	-1.62089			
22	-2.57750	-2.18852				
23	-3.03362					

	7	8	9	10	11	12
7	0.51362					
8	0.41726	0.35611				
9	0.32451	0.28054	0.23826			
10	0.23430	0.20722	0.18131	0.15636		
11	0.14580	0.13543	0.12573	0.11662	0.10808	
12	0.05828	0.06457	0.07101	0.07769	0.08470	0.09212
13	-0.02889	-0.00590	0.01673	0.03923	0.06182	
14	-0.11631	-0.07649	-0.03754	0.00093		
15	-0.20464	-0.14773	-0.09221			
16	-0.29458	-0.22020				
17	-0.38697					

TRIANGULAR:

	1	2	3	4	5	6
1	0.85350					
2	0.70229	0.59275				
3	0.59097	0.49809	0.42843			
4	0.49899	0.42040	0.36146	0.31234		
5	0.41891	0.35301	0.30360	0.26241	0.22638	
6	0.34705	0.29271	0.25195	0.21799	0.18827	0.16152
7	0.28131	0.23764	0.20488	0.17758	0.15369	0.13219
8	0.22030	0.18659	0.16131	0.14024	0.12181	0.10522
9	0.16297	0.13869	0.12048	0.10531	0.09204	0.08009
10	0.10844	0.09320	0.08176	0.07223	0.06389	0.05639
11	0.05587	0.04938	0.04451	0.04046	0.03691	0.03372
12	0.00432	0.00648	0.00810	0.00945	0.01063	0.01169
13	-0.04715	-0.03631	-0.02817	-0.02140	-0.01547	-0.01012
14	-0.09950	-0.07978	-0.06499	-0.05267	-0.04188	-0.03217
15	-0.15365	-0.12472	-0.10302	-0.08494	-0.06912	-0.05487
16	-0.21048	-0.17186	-0.14290	-0.11876	-0.09764	-0.07863
17	-0.27084	-0.22193	-0.18524	-0.15467	-0.12792	-0.10384
18	-0.33576	-0.27576	-0.23077	-0.19327	-0.16046	-0.13093
19	-0.40654	-0.33446	-0.28041	-0.23536	-0.19594	
20	-0.48517	-0.39968	-0.33556	-0.28212		
21	-0.57504	-0.47421	-0.39858			
22	-0.68288	-0.56364				
23	-0.82667					

	7	8	9	10	11	12
7	0.11249					
8	0.09002	0.07592				
9	0.06915	0.05900	0.04951			
10	0.04952	0.04315	0.03723	0.03170		
11	0.03081	0.02812	0.02565	0.02341	0.02143	
12	0.01268	0.01362	0.01456	0.01556	0.01670	0.01807
13	-0.00522	-0.00063	0.00373	0.00799	0.01227	
14	-0.02326	-0.01495	-0.00708	0.00051		
15	-0.04180	-0.02963	-0.01813			
16	-0.06119	-0.04495				
17	-0.08175					

U-SHAPED:

	1	2	3	4	5	6
1	1.57129					
2	1.52031	1.47274				
3	1.46528	1.41958	1.37049			
4	1.40504	1.36142	1.31460	1.26387		
5	1.33735	1.29613	1.25195	1.20421	1.15204	
6	1.25791	1.21963	1.17869	1.13459	1.08678	1.03429
7	1.15870	1.12423	1.08747	1.04808	1.00580	0.96044
8	1.02738	0.99801	0.96684	0.93370	0.89863	0.86210
9	0.84952	0.82690	0.80309	0.77812	0.75232	0.72676
10	0.61506	0.60092	0.58630	0.57142	0.55693	0.54444
11	0.32685	0.32250	0.31844	0.31506	0.31329	0.31525
12	0.00512	0.01100	0.01792	0.02643	0.03770	0.05427
13	-0.31713	-0.30160	-0.28437	-0.26477	-0.24139	-0.21139
14	-0.60670	-0.58292	-0.55691	-0.52789	-0.49428	-0.45296
15	-0.84289	-0.81261	-0.77969	-0.74326	-0.70163	-0.65142
16	-1.02239	-0.98724	-0.94913	-0.90713	-0.85945	-0.80254
17	-1.15497	-1.11617	-1.07418	-1.02800	-0.97577	-0.91376
18	-1.25502	-1.21339	-1.16837	-1.11893	-1.06312	-0.99707
19	-1.33498	-1.29101	-1.24348	-1.19134	-1.13254	
20	-1.40298	-1.35697	-1.30727	-1.25275		
21	-1.46343	-1.41558	-1.36391			
22	-1.51861	-1.46908				
23	-1.56971					

	7	8	9	10	11	12
7	0.91089					
8	0.82475	0.78471				
9	0.70341	0.68369	0.66313			
10	0.53698	0.53823	0.54889	0.55874		
11	0.32485	0.34743	0.38687	0.43969	0.48709	
12	0.08078	0.12387	0.18961	0.27815	0.37779	0.46145
13	-0.16950	-0.10807	-0.01943	0.09887	0.23835	
14	-0.39819	-0.32149	-0.21397	-0.07161		
15	-0.58652	-0.49778	-0.37532			
16	-0.72992	-0.63189				
17	-0.83521					

LAPLACIAN:

	1	2	3	4	5	6
1	10.85095					
2	5.51743	4.76866				
3	4.15467	3.45595	2.72752			
4	3.24794	2.67774	2.10175	1.70009		
5	2.57255	2.11219	1.65426	1.33562	1.09618	
6	2.03539	1.66884	1.30617	1.05405	0.86462	0.71307
7	1.58962	1.30441	1.02153	0.82490	0.67717	0.55899
8	1.20822	0.99476	0.78063	0.63168	0.51977	0.43024
9	0.87360	0.72464	0.57117	0.46421	0.38384	0.31956
10	0.57267	0.48301	0.38439	0.31534	0.26345	0.22194
11	0.29423	0.26065	0.21307	0.17925	0.15381	0.13348
12	0.02757	0.04890	0.05049	0.05055	0.05055	0.05058
13	-0.23805	-0.16090	-0.11007	-0.07613	-0.05068	-0.03030
14	-0.51337	-0.37742	-0.27533	-0.20616	-0.15425	-0.11270
15	-0.80913	-0.60931	-0.45200	-0.34490	-0.26452	-0.20019
16	-1.13639	-0.86548	-0.64696	-0.49784	-0.38592	-0.29635
17	-1.50785	-1.15600	-0.86796	-0.67113	-0.52338	-0.40514
18	-1.94018	-1.49402	-1.12504	-0.87266	-0.68321	-0.53161
19	-2.45861	-1.89932	-1.43328	-1.11429	-0.87483	
20	-3.1[illegible]53	-2.40584	-1.81849	-1.41624		
21	-3.96958	-3.08056	-2.33161			
22	-5.22792	-4.06494				
23	-6.70369					

	7	8	9	10	11	12
7	0.46052					
8	0.35567	0.29181				
9	0.26602	0.22022	0.18032			
10	0.18740	0.15790	0.13233	0.11005		
11	0.11658	0.10223	0.08998	0.07967	0.07135	
12	0.05065	0.05088	0.05144	0.05267	0.05502	0.05900
13	-0.01325	-0.00155	0.01495	0.02774	0.04079	
14	-0.07800	-0.04804	-0.02130	0.00353		
15	-0.14649	-0.10020	-0.05909			
16	-0.22160	-0.15723				
17	-0.30650					

NORMAL:

	1	2	3	4	5	6
1	3.98637					
2	2.98631	2.34579				
3	2.43002	1.90210	1.59034			
4	2.02212	1.58060	1.31974	1.12496		
5	1.68845	1.31934	1.10125	0.93840	0.80441	
6	1.39882	1.09350	0.91316	0.77855	0.66782	0.57172
7	1.13775	0.89051	0.74459	0.63575	0.54628	0.46870
8	0.89606	0.70298	0.58919	0.50442	0.43484	0.37458
9	0.66772	0.52610	0.44285	0.38098	0.33032	0.28656
10	0.44841	0.35642	0.30265	0.26289	0.23051	0.20270
11	0.23476	0.19129	0.16635	0.14823	0.13375	0.12155
12	0.02397	0.02850	0.03210	0.03541	0.03864	0.04193
13	-0.18655	-0.13396	-0.10179	-0.07702	-0.05602	-0.03722
14	-0.39934	-0.29808	-0.23695	-0.19043	-0.15143	-0.11691
15	-0.61714	-0.46597	-0.37515	-0.30632	-0.24885	-0.19819
16	-0.84317	-0.64013	-0.51844	-0.42641	-0.34973	-0.28228
17	-1.08151	-0.82370	-0.66940	-0.55287	-0.45589	-0.37070
18	-1.33774	-1.02097	-0.83158	-0.68866	-0.56982	-0.46552
19	-1.62022	-1.23838	-1.01024	-0.83819	-0.69523	-0.56982
20	-1.94276	-1.48654	-1.21411	-1.00875	-0.83819	
21	-2.33148	-1.78554	-1.45966	-1.21411		
22	-2.84768	-2.18247				
23	-3.71308					

	7	8	9	10	11	12
7	0.39906					
8	0.32057	0.27100				
9	0.24745	0.21167	0.17836			
10	0.17801	0.15557	0.13485	0.11549		
11	0.11097	0.10162	0.09325	0.08573	0.07899	
12	0.04534	0.04895	0.05284	0.05707	0.06176	0.06701
13	-0.01978	-0.00316	0.01301	0.02904	0.04520	
14	-0.08524	-0.05543	-0.02678	-0.00123		
15	-0.15191	-0.10855	-0.06709			
16	-0.22081	-0.16335				
17	-0.29317					

TABLE XXV

Correlation matrices of the order statistics of the additive noise for a filter length of twenty-five, for six parent distributions.

UNIFORM:

	1	2	3	4	5
1	2.57265				
2	2.35897	2.17948			
3	2.14530	1.98290	1.82051		
4	1.93162	1.78632	1.64102	1.49573	
5	1.71795	1.58974	1.46154	1.33333	1.20513
6	1.50427	1.39316	1.28205	1.17094	1.05983
7	1.29060	1.19658	1.10256	1.00855	0.91453
8	1.07692	1.00000	0.92308	0.84615	0.76923
9	0.86325	0.80342	0.74359	0.68376	0.62393
10	0.64957	0.60684	0.56410	0.52137	0.47863
11	0.43590	0.41026	0.38462	0.35897	0.33333
12	0.22222	0.21368	0.20513	0.19658	0.18803
13	0.00855	0.01709	0.02564	0.03419	0.04274
14	-0.20513	-0.17949	-0.15385	-0.12821	-0.10256
15	-0.41880	-0.37607	-0.33333	-0.29060	-0.24786
16	-0.63248	-0.57265	-0.51282	-0.45299	-0.39316
17	-0.84615	-0.76923	-0.69231	-0.61538	-0.53846
18	-1.05983	-0.96581	-0.87179	-0.77778	-0.68376
19	-1.27350	-1.16239	-1.05128	-0.94017	-0.82906
20	-1.48718	-1.35897	-1.23077	-1.10256	-0.97436
21	-1.70086	-1.55555	-1.41025	-1.26496	-1.11966
22	-1.91453	-1.75214	-1.58974	-1.42735	
23	-2.12820	-1.94872	-1.76923		
24	-2.34188	-2.14530			
25	-2.55555				

	6	7	8	9	10
6	0.94872				
7	0.82051	0.72650			
8	0.69231	0.61538	0.53846		
9	0.56410	0.50427	0.44444	0.38462	
10	0.43590	0.39316	0.35043	0.30769	0.26496
11	0.30769	0.28205	0.25641	0.23077	0.20513
12	0.17949	0.17094	0.16239	0.15385	0.14530
13	0.05128	0.05983	0.06838	0.07692	0.08547
14	-0.07692	-0.05128	-0.02564	0.00000	0.02564
15	-0.20513	-0.16239	-0.11966	-0.07692	-0.03419
16	-0.33333	-0.27350	-0.21368	-0.15385	-0.09402
17	-0.46154	-0.38462	-0.30769	-0.23077	
18	-0.58974	-0.49573	-0.40171		
19	-0.71795	-0.60684			
20	-0.84615				

	11	12	13
11	0.17949		
12	0.13675	0.12821	
13	0.09402	0.10256	0.11111
14	0.05128	0.07692	
15	0.00855		

PARABOLIC:

	1	2	3	4	5
1	3.17216				
2	2.71519	2.36344			
3	2.36372	2.05577	1.81621		
4	2.06239	1.79334	1.58404	1.40394	
5	1.79094	1.55763	1.37614	1.21999	1.07924
6	1.53936	1.33957	1.18420	1.05054	0.93009
7	1.30177	1.13392	1.00343	0.89121	0.79010
8	1.07436	0.93729	0.83076	0.73919	0.65673
9	0.85444	0.74727	0.66404	0.59255	0.52821
10	0.63998	0.56207	0.50165	0.44982	0.40324
11	0.42935	0.38027	0.34232	0.30986	0.28077
12	0.22117	0.20066	0.18498	0.17172	0.15997
13	0.01423	0.02218	0.02868	0.03454	0.04008
14	-0.19264	-0.15619	-0.12747	-0.10245	-0.07961
15	-0.40059	-0.33544	-0.28435	-0.24004	-0.19978
16	-0.61083	-0.51663	-0.44288	-0.37905	-0.32114
17	-0.82471	-0.70092	-0.60410	-0.52037	-0.44449
18	-1.04382	-0.88968	-0.76920	-0.66506	-0.57075
19	-1.27013	-1.08461	-0.93967	-0.81444	-0.70106
20	-1.50624	-1.28796	-1.11747	-0.97020	-0.83692
21	-1.75580	-1.50286	-1.30534	-1.13477	-0.98042
22	-2.02436	-1.73410	-1.50748	-1.31180	
23	-2.32133	-1.98976	-1.73093		
24	-2.66539	-2.28593			
25	-3.10625				

	6	7	8	9	10
6	0.81856				
7	0.69652	0.60836			
8	0.58045	0.50862	0.44012		
9	0.46875	0.41280	0.35950	0.30825	
10	0.36024	0.31985	0.28144	0.24457	0.20893
11	0.25401	0.22896	0.20523	0.18254	0.16071
12	0.14930	0.13945	0.13027	0.12164	0.11352
13	0.04544	0.05074	0.05606	0.06145	0.06699
14	-0.05819	-0.03771	-0.01786	0.00158	0.02080
15	-0.16218	-0.12642	-0.09194	-0.05835	-0.02534
16	-0.26716	-0.21592	-0.16663	-0.11872	-0.07174
17	-0.37383	-0.30682	-0.24243	-0.17992	
18	-0.48297	-0.39979	-0.31991		
19	-0.59559	-0.49568			
20	-0.71296				

	11	12	13
11	0.13961		
12	0.10586	0.09863	
13	0.07272	0.07873	0.08507
14	0.03996	0.05922	
15	0.00735		

TRIANGULAR:

	1	2	3	4	5
1	0.87445				
2	0.72767	0.61936			
3	0.61952	0.52659	0.45689		
4	0.53012	0.45038	0.39057	0.34073	
5	0.45226	0.38424	0.33323	0.29072	0.25352
6	0.38240	0.32504	0.28202	0.24617	0.21480
7	0.31849	0.27097	0.23532	0.20562	0.17963
8	0.25921	0.22088	0.19213	0.16817	0.14721
9	0.20362	0.17396	0.15171	0.13317	0.11695
10	0.15097	0.12956	0.11350	0.10012	0.08841
11	0.10059	0.08712	0.07702	0.06860	0.06123
12	0.05179	0.04605	0.04175	0.03817	0.03503
13	0.00382	0.00573	0.00716	0.00835	0.00940
14	-0.04409	-0.03451	-0.02732	-0.02133	-0.01608
15	-0.09272	-0.07531	-0.06225	-0.05137	-0.04185
16	-0.14281	-0.11732	-0.09820	-0.08227	-0.06833
17	-0.19507	-0.16113	-0.13567	-0.11446	-0.09590
18	-0.25016	-0.20731	-0.17517	-0.14838	-0.12495
19	-0.30883	-0.25648	-0.21721	-0.18449	-0.15586
20	-0.37197	-0.30940	-0.26247	-0.22336	-0.18914
21	-0.44085	-0.36712	-0.31182	-0.26575	-0.22543
22	-0.51737	-0.43125	-0.36666	-0.31284	
23	-0.60482	-0.50454	-0.42934		
24	-0.70976	-0.59249			
25	-0.84968				

	6	7	8	9	10
6	0.18656				
7	0.15625	0.13481			
8	0.12834	0.11105	0.09500		
9	0.10235	0.08896	0.07654	0.06491	
10	0.07788	0.06822	0.05926	0.05087	0.04298
11	0.05460	0.04853	0.04290	0.03763	0.03270
12	0.03221	0.02962	0.02723	0.02501	0.02296
13	0.01034	0.01120	0.01202	0.01281	0.01360
14	-0.01137	-0.00704	-0.00301	0.00080	0.00446
15	-0.03328	-0.02542	-0.01811	-0.01123	-0.00465
16	-0.05578	-0.04427	-0.03357	-0.02351	-0.01392
17	-0.07919	-0.06388	-0.04964	-0.03625	
18	-0.10385	-0.08451	-0.06653		
19	-0.13010	-0.10647			
20	-0.15834				

	11	12	13
11	0.02807		
12	0.02109	0.01943	
13	0.01446	0.01543	0.01660
14	0.00805	0.01167	
15	0.00173		

U-SHAPED:

	1	2	3	4	5
1	1.57884				
2	1.53219	1.48836			
3	1.48224	1.43995	1.39484		
4	1.42828	1.38765	1.34434	1.29778	
5	1.36893	1.33017	1.28888	1.24455	1.19654
6	1.30183	1.26526	1.22633	1.18461	1.13957
7	1.22233	1.18844	1.15242	1.11392	1.07254
8	1.12248	1.09206	1.05981	1.02548	0.98880
9	0.99077	0.96496	0.93770	0.90885	0.87833
10	0.81471	0.79491	0.77415	0.75238	0.72973
11	0.58659	0.57423	0.56146	0.54838	0.53527
12	0.31029	0.30644	0.30279	0.29952	0.29713
13	0.00444	0.00948	0.01528	0.02217	0.03078
14	-0.30183	-0.28838	-0.27364	-0.25721	-0.23832
15	-0.57923	-0.55852	-0.53608	-0.51145	-0.48375
16	-0.80880	-0.78226	-0.75367	-0.72249	-0.68777
17	-0.98625	-0.95528	-0.92200	-0.88583	-0.84576
18	-1.11907	-1.08476	-1.04794	-1.00800	-0.96387
19	-1.21969	-1.18279	-1.14322	-1.10034	-1.05303
20	-1.29968	-1.26064	-1.21882	-1.17352	-1.12359
21	-1.36708	-1.32621	-1.28242	-1.23504	-1.18283
22	-1.42662	-1.38411	-1.33857	-1.28930	
23	-1.48073	-1.43670	-1.38956		
24	-1.53077	-1.48535			
25	-1.57752				

	6	7	8	9	10
6	1.09039				
7	1.02783	0.97886			
8	0.94966	0.90799	0.86259		
9	0.84632	0.81346	0.78034	0.74476	
10	0.70669	0.68451	0.66508	0.64942	0.63252
11	0.52295	0.51319	0.50895	0.51335	0.52625
12	0.29669	0.30047	0.31227	0.33678	0.37674
13	0.04243	0.05978	0.08736	0.13106	0.19559
14	-0.21544	-0.18557	-0.14363	-0.08277	0.00317
15	-0.45128	-0.41079	-0.35670	-0.28142	-0.17769
16	-0.64768	-0.59872	-0.53495	-0.44815	-0.33018
17	-0.79984	-0.74440	-0.67315	-0.57734	
18	-0.91352	-0.85310	-0.77605		
19	-0.99920	-0.93483			
20	-1.06687				

	11	12	13
11	0.53727		
12	0.42790	0.47251	
13	0.27993	0.37269	0.44947
14	0.11520	0.24474	
15	-0.04314		

LAPLACIAN:

	1	2	3	4	5
1	11.35767				
2	5.79685	5.11205			
3	4.42470	3.74950	2.98924		
4	3.50842	2.94634	2.33594	1.90736	
5	2.82473	2.36167	1.86812	1.52247	1.26263
6	2.28047	1.90289	1.50376	1.22458	1.01473
7	1.82866	1.52563	1.20562	0.98187	0.81369
8	1.44238	1.20525	0.95334	0.77715	0.64473
9	1.10450	0.92648	0.73445	0.60000	0.49893
10	0.80292	0.67880	0.54045	0.44336	0.37038
11	0.52803	0.45402	0.36485	0.30192	0.25460
12	0.27111	0.24493	0.20193	0.17103	0.14777
13	0.02363	0.04449	0.04618	0.04625	0.04625
14	-0.22307	-0.15440	-0.10795	-0.07691	-0.05365
15	-0.47767	-0.35888	-0.26606	-0.20298	-0.15565
16	-0.74874	-0.57601	-0.43369	-0.33643	-0.26344
17	-1.04495	-0.81289	-0.61639	-0.48176	-0.38069
18	-1.37573	-1.07719	-0.82016	-0.64376	-0.51132
19	-1.75271	-1.37832	-1.05226	-0.82825	-0.66006
20	-2.19213	-1.72927	-1.32275	-1.04324	-0.83338
21	-2.71931	-2.15029	-1.64725	-1.30116	-1.04129
22	-3.37823	-2.67654	-2.05284	-1.62352	
23	-4.25568	-3.37734	-2.59297		
24	-5.52850	-4.39458			
25	-6.93654				

	6	7	8	9	10
6	0.84684				
7	0.67915	0.56703			
8	0.53878	0.45050	0.37486		
9	0.41808	0.35072	0.29300	0.24257	
10	0.31199	0.26334	0.22168	0.18531	0.15315
11	0.21674	0.18521	0.15822	0.13472	0.11405
12	0.12917	0.11369	0.10047	0.08903	0.07914
13	0.04626	0.04629	0.04637	0.04660	0.04714
14	-0.03504	-0.01950	-0.00611	0.00579	0.01676
15	-0.11778	-0.08620	-0.05905	-0.03507	-0.01329
16	-0.20503	-0.15633	-0.11450	-0.07764	-0.04434
17	-0.29982	-0.23240	-0.17450	-0.12356	
18	-0.40536	-0.31703	-0.24118		
19	-0.52550	-0.41332			
20	-0.66547				

	11	12	13
11	0.09584		
12	0.07074	0.06392	
13	0.04825	0.05030	0.05368
14	0.02740	0.03835	
15	0.00722		

NORMAL:

	1	2	3	4	5
1	4.12097				
2	3.12090	2.47027			
3	2.56653	2.02466	1.70597		
4	2.16159	1.70278	1.43280	1.23189	
5	1.83169	1.44217	1.21291	1.04230	0.90236
6	1.54660	1.21782	1.02434	0.88037	0.76231
7	1.29084	1.01709	0.85607	0.73629	0.63812
8	1.05531	0.83260	0.70170	0.60441	0.52471
9	0.83408	0.65957	0.55714	0.48110	0.41889
10	0.62298	0.49466	0.41952	0.36387	0.31844
11	0.41887	0.33535	0.28670	0.25085	0.22172
12	0.21919	0.17962	0.15698	0.14055	0.12743
13	0.02175	0.02574	0.02887	0.03172	0.03447
14	-0.17547	-0.12788	-0.09895	-0.07681	-0.05815
15	-0.37449	-0.28283	-0.22781	-0.18616	-0.15141
16	-0.57746	-0.44079	-0.35912	-0.29753	-0.24635
17	-0.78683	-0.60367	-0.49447	-0.41227	-0.34411
18	-1.00560	-0.77381	-0.63580	-0.53204	-0.44610
19	-1.23768	-0.95424	-0.78564	-0.65898	-0.55416
20	-1.48857	-1.14925	-0.94753	-0.79609	-0.67082
21	-1.76657	-1.36526	-1.12631	-0.94788	-0.79993
22	-2.08548	-1.61301	-1.33239	-1.12188	
23	-2.47160	-1.91290	-1.58117		
24	-2.98671	-2.31288			
25	-3.85475				

	6	7	8	9	10
6	0.66016				
7	0.55320	0.47719			
8	0.45583	0.39423	0.33780		
9	0.36520	0.31724	0.27338	0.23254	
10	0.27932	0.24447	0.21269	0.18318	0.15543
11	0.19677	0.17466	0.15462	0.13614	0.11891
12	0.11639	0.10680	0.09831	0.09068	0.08378
13	0.03724	0.04007	0.04304	0.04618	0.04956
14	-0.04156	-0.02627	-0.01182	0.00212	0.01580
15	-0.12083	-0.09294	-0.06687	-0.04200	-0.01789
16	-0.20146	-0.16070	-0.12273	-0.08668	-0.05190
17	-0.28444	-0.23037	-0.18011	-0.13250	
18	-0.37097	-0.30295	-0.23983		
19	-0.46258	-0.37976			
20	-0.56145				

	11	12	13
11	0.10267		
12	0.07753	0.07187	
13	0.05322	0.05726	0.06175
14	0.02942	0.04317	
15	0.00581		